Charity Auctions

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Abstract

In a charity auction, the public-goods nature of auction revenue affects bidding incentives. We compare equilibrium bidding and revenue in first-price, second-price, and all-pay charity auctions. Bidding revenue typically varies by selling format. First-price auctions are less lucrative than second-price and all-pay auctions, and with sufficiently many bidders the all-pay auction has the highest bidding revenue. However, revenue equivalence applies when the auctioneer can set a reserve price and fees plus threaten to dissolve the auction. If the auctioneer cannot threaten dissolution, a reserve and bidding fee can augment revenue but again revenue varies by auction format.

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1 Introduction

In a typical auction the only bidder who benefits from a sale is the winner, and bidders regard any payments as pure losses. A charity auction, in which an item is sold to raise revenue for a charitable cause, is different. When an auction is held to benefit cancer research or a school, it is reasonable to assume that bidders have two objectives: to win items that they value and to support the charitable cause. If this is the case, then each dollar raised at the auction provides a benefit to auction participants. Auction revenue may be viewed as a public good that is beneficial to all bidders regardless of the revenue's source. This raises interesting questions about bidding strategies and revenue in charity auctions. Do bids rise because they are "subsidized" by charitable sentiment, or do they fall because there are benefits from losing a charity auction? Do common auction formats differ in their expected revenue, or does revenue equivalence apply to charity auctions?

Charity auctions are a common and lucrative way of raising funds. One of the longest-running auctions in the world, the annual wine sale hosted by the Hospices de Beaune, is a charity auction. The 2005 sale, the 145th auction organized by the Hospices, benefitted Burgundy-area charitable groups and raised almost $6 million in revenue. An American counterpart of this auction is the annual Auction Napa Valley wine sale, which was held for the 25th time in 2005 and generated $10.5 million in revenue. Musician Eric Clapton conducted a charity auction in 1999 that offered an ironic complement to these wine sales. Clapton, who struggled with alcohol and drug addiction during the 1970s, sold 100 of his guitars and raised $5 million for his substance abuse treatment facility on the Caribbean island of Antigua. Charity auctions are frequently used to raise funds for schools but these auctions usually do not collect extraordinary amounts of revenue. An exception is documented by David Kaplan (1999) in *The Silicon Boys and Their Valley of Dreams.* Woodside, California is a suburb of San Francisco that has a large population of computer industry multi-millionaires. When the region's public elementary school held a charity auction in 1998 the prizes included a week at NASA's Space Camp and a week-long cruise on Oracle founder Larry Ellison's yacht; over $400,000 was raised in a single night.

We present a model of a charity auction in which risk-neutral bidders have independently drawn private values for a single auctioned item and each bidder receives a benefit from the charity's revenue. We permit bidders to benefit more strongly from their own payments than from those of other bidders, but the model is symmetric in all other respects. We consider three auction formats:

first-price, second-price, and all-pay.[1] Although there are more complicated auction formats that have higher expected revenue in some settings, we focus on these three relatively simple auctions because their bidding rules are easy to implement, and similar auctions are frequently observed in actual fund-raising activity. For these auction formats, we consider how revenue varies when the auctioneer is able to charge the bidders fees and set a reserve price. We also discuss the revenue and utility implications if potential bidders can opt out of an auction.

We find that bidding in a first-price charity auction is higher than in a standard (non-charity) auction because of the benefit winners receive from their own payments. A similar effect exists in second-price auctions, plus there is the possibility that a bidder submits the second-highest tender and determines the payment of the winner. This additional incentive to increase one's bid in a second-price charity auction unbalances the revenue equivalence result first presented by Vickrey (1961) and later generalized by Myerson (1981). In an *absolute* auction – one without a reserve price or bidding fees – a second-price charity auction is more lucrative than a first-price charity auction.

When a bidder in a first- or second-price auction increases her bid, she decreases the probability that another bidder will win the auction and make a payment to the auctioneer. In a charity auction other bidders' payments to the auctioneer are valuable, and this may depress bids. All-pay charity auctions do not have this characteristic; increasing one's bid in an all-pay auction does not affect the expected payments of other bidders. We demonstrate that the expected revenue in an all-pay absolute auction is greater than in either single-payer auction when the number of bidders is large. Although revenue in an all-pay absolute auction is always greater than in a first-price absolute auction, it is not possible to provide a similar revenue ranking for all-pay and second-price auctions.

We consider two settings for the use of fees and minimum bids to increase revenue. In the first, the auctioneer can threaten to dissolve the contest if he does not receive payments from all potential participants. This is called the *strong auctioneer* case below. We demonstrate that in this situation there are many revenue-equivalent auction formats that extract maximal surplus from the bidders. While this result provides an interesting benchmark for auction design problems, it seems unlikely that an auctioneer would be able to design and implement such a mechanism. Therefore we also consider a *weak auctioneer* case in which the auctioneer cannot shut down the auction if any bidders fail to participate, so bidders can opt out of the auction and still receive a

[1] In an all-pay auction, all bidders pay their bids and the highest bidder receives the item for sale.

positive benefit from the payments of active bidders. The ability of non-participants to free-ride is similar to the voluntary participation problem that is central to the literature on financing public goods, and therefore important to the design of fund-raising auctions. We rank expected revenue with a weak auctioneer in a simple two-bidder auction and when the number of bidders is large, and show that the ranking of auction formats is similar to the absolute auction case. While revenue and utility may be unbounded when the auctioneer is strong, this is not the case for a weak auctioneer.

This paper is part of a growing literature on auctions with revenue-dependent externalities among bidders. One of the first papers in this area is Engelbrecht-Wiggans' (1994) study of first- and second-price auctions with benefits to bidders that are proportional to the winning bid. The motivating example used by Engelbrecht-Wiggans is an estate auction in which bidders (the children of the deceased) assign ownership of a family farm to the highest bidder, but all bidders collect an equal share of the auction revenue. We extend Engelbrecht-Wiggans' model by allowing a more general structure of benefits to auction participants, considering all-pay auctions in addition to first- and second-price formats, and studying how these auction formats do (or do not) function as optimal revenue-raising mechanisms. Further analysis of first- and second-price auctions extending the Engelbrecht-Wiggans model include Ettinger (2002) and Morgan, Steiglitz, and Reis (2003).[2] Revenue-proportional externalities are also relevant to corporate buy-outs, in which a bidder with partial ownership of the firm for sale (a "toehold") receives a portion of the winner's payment. Burkart (1995) and Singh (1998) consider the effect of toeholds on bidding in a private-value auction; Bulow, Huang, and Klemperer (1999) demonstrate that bidders' incentives and sale prices in common-value takeover auctions can be substantially different from the private-value case.

The paper most closely related to ours is by Goeree, Maasland, Onderstal, and Turner (2005), who also consider auctions in which revenue is regarded as a public good. Their model is equivalent our charity auction model, with an added restriction that bidders' utility from revenue is invariant to the revenue's source. Goeree *et al.* demonstrate that an auction in which all bidders pay the lowest bid has greater revenue from bids than any "winner-pay" contest such as a first- or second-price auction. They also show that their proposed auction format is revenue-maximizing when a strong auctioneer can set a fee, a reservation price, and also threaten to dissolve the contest.

[2] Ettinger (2002) considers the difference between identity-dependent and identity-independent price externalities, with charity auctions as an example of the latter type of externality. Morgan, Steiglitz, and Reis (2003) study the effect of spiteful motives among bidders as a possible explanation for laboratory experiments that does not match behavior predicted in standard auction models.

However, we show that when the relatively common winner-pay auctions are associated with the appropriate fees, these auctions are also optimal. An additional differentiating feature of the present paper is our consideration of a weak auctioneer who can set fees and a reserve but cannot cancel the auction to deter non-participation.

Other than auctions with revenue-proportional benefits to bidders, there are several other settings in which externalities or transfers among bidders are important. "Knockout" auctions that determine allocations among members of a cartel typically include a payment to auction participants who do not win the knockout auction. Graham and Marshall (1987) and McAfee and McMillan (1992) characterize optimal mechanisms for allocating auction profits among members of a cartel.[3] Jehiel, Moldovanu, and Stacchetti (1996, 1999) consider situations in which externalities exist among bidders through their valuations of auction outcomes in which they do not receive the object. These externalities differ from charity auctions in that their magnitudes are independent of bidding intensity. Budget-constrained bidders may also impose externalities on each other when participating in a sequence of auctions. Bidding is aggressive in an early auction because a high price weakens the ability of the early winner to succeed in later auctions. Pitchik and Schotter (1988) and Benoît and Krishna (2001) study this phenomenon for the case of common values over the auctioned items and complete information about valuations and budgets.

The description of auction revenue as a public good invites comparison between the present research and other studies of funding public goods. When contributions to a pure public good are voluntary, contributions from one source crowd out other donations. Charity auctions are similar in that bidders may depress their bids when the public goods effect from others' payments becomes more valuable. However, personal gains from winning the object limit crowding out in our model, and similar results on contributions to public goods are discussed by Andreoni (1989) and Morgan (2000). Andreoni describes how giving to a public good increases when donors receive an egotistical "warm glow" from their own contributions. Similarly, we find that a warm glow can increase auction revenue. Morgan studies the use of lotteries to raise revenue for a public good. As in an auction, all lottery participants have a chance to win a prize for personal gain, and the

[3] Both papers include payments to losing participants in a knockout auction, but the mechanisms do not provide each bidder with a positive benefit that is proportional to the amount paid (as in a charity auction). In Graham and Marshall (1987) all n participants in the knockout auction receive the same payment *before* bids are submitted in the knockout round. In McAfee and McMillan (1992) the losers of the knockout auction each get $\frac{1}{n-1}$ of the difference between the winner's payment and a reservation price; the winner's utility is the difference between her valuation and her payment.

combination of private and public gains from ticket purchases increases revenue relative to the case with voluntary donations directly to the public good. A key difference between Morgan's lottery model and a charity auction is that all of the lottery contestants value the prize equally, so there are no (in)efficiency consequences of an allocation mechanism that is random.[4]

2 Preliminaries

Suppose an auctioneer possesses one item to sell at a charity auction, and all revenue from the sale will benefit the charitable cause. There are $n \geq 2$ risk-neutral bidders who draw independently from the probability distribution F with support $[\underline{t}, \overline{t}]$ to determine their private valuations for the auctioned item. An individual's taste, t, for the item is private information, but the distribution F is common knowledge. Let F be differentiable on the interior of its support with positive density f. When n values of t are drawn from F, let F_k^n denote the distribution of the k^{th}-highest order statistic. If $k > n$, we use the notational convention $F_k^n = 0$.

Individual surplus from the charity auction has both private and philanthropic components. As in the standard independent private values auction model, an individual receives surplus from private consumption when she purchases the object at a price less than her valuation. Additional surplus comes to the individual through the charity collecting revenue for its cause. We specify that θ is the return to bidder i from the charity collecting one dollar from i, while λ is the return to i when the charity receives another bidder's dollar. The parameters satisfy $0 \leq \lambda \leq \theta < 1$. The parameter θ is less than one because values of $\theta \geq 1$ render the idea of a charity auction moot since i is willing to transfer any amount of money to the charity. If $\theta = \lambda$, the return from the charity's revenue is purely altruistic; individual i's satisfaction from seeing money go to a favored cause is independent of the revenue's source. When $\theta > \lambda$, there is an additional, egotistical benefit to bidder i when her own money is received by the charity, as in Andreoni (1989). This warm glow is denoted $\Delta = \theta - \lambda$.

The charitable organization considers three sealed-bid auction formats: first-price, second-price, and all-pay. For each auction format, we consider bidding and revenue with and without a reserve price (r) and fixed fees. We use the term *absolute auction* to refer to the case in which the auctioneer sets no reserve or fees. To keep the bidders' and auctioneer's strategies as simple as

[4] The lottery contestant who purchases the most tickets has the best chance of winning the prize, but other contestants may win instead.

possible, we assume that bidders take their participation and bidding actions simultaneously. Each of the n potential bidders submits a *bidding message* which may be "no bid" or a nonnegative value b which must be no smaller than r in auctions with a reserve. When the auctioneer can charge fees, we allow the auctioneer to set two fees that are paid (or refused) at the same time the bidding message is sent. All n bidders, including those who announce "no bid" may be asked to pay the *continuation fee* c. Additionally, bidders who submit bids in $[r, \infty)$ may be asked to pay the *bidding fee* φ. (The auctioneer ignores otherwise valid bids without the necessary payment of φ.) A bidder who pays φ and submits a bid at or above r is called *active*. The auctioneer's ability to respond to bidders' messages and payments depends on whether we assume that he is *strong* or *weak*. These properties are defined:

Definition 1 *A* ***strong auctioneer*** *has the power to dissolve a charity auction unless each of the n potential bidders pays the continuation fee, c. If any bidder refrains from paying c, the strong auctioneer ends the auction without awarding the object or collecting payments from any bidder.*

Definition 2 *A* ***weak auctioneer*** *is unable to dissolve the auction just because one or more bidders declines to pay c. If at least one bidder pays the bidding fee φ and announces a bid above the reserve, the weak auctioneer must award the prize and collect payments from active bidders.*

The absolute auction may be regarded as a special case in the weak auctioneer framework. If any potential bidder opts out of an auction of this sort, the auctioneer awards the item to the high bidder and collects revenue from valid bids according to the format of the auction. Regardless of the type of auctioneer or restrictions on r, φ, and c, if a bidder opts out of an auction she receives utility of λ times the revenue collected.

Our consideration of revenue-maximizing strategies requires a restriction on parameter values when the auctioneer is strong. In this case we assume $\Omega \equiv [1 - \theta - \lambda(n-1)] > 0$. If this inequality does not hold, the bidders – who do not have budget constraints – could all be asked to transfer any amount of money to the auctioneer under the threat that if any bidder refuses the mechanism will be dissolved. Under these circumstances, all bidders would make unlimited transfers to the auctioneer, and the process could generate both infinite revenue for the auctioneer and infinite utility for the bidders. While we avoid this situation with a restriction on parameter values, there are other assumptions that would have a similar effect.[5] We show below that no such restrictions

[5] Goeree *et al.* (2005) assume that bidders are budget constrained, so that when $\Omega < 0$ the bidders simply transfer all of their money to the auctioneer.

are necessary in the absolute auction or when the auctioneer is weak.

3 Bidding and Utility

In this section we derive bidders' equilibrium strategies in auctions with and without a reserve price or fees. For each auction format we consider, we begin with the absolute auction case, when $r = \varphi = c = 0$. We derive equilibrium bid functions by considering the signalling problem of bidder i given that all other bidders are active and use the same increasing, differentiable function B to map their own valuations of the auctioned item into bids.[6] Bidder i is not obliged to implicitly announce her true type, t. The bidder selects a valuation s from $[\underline{t}, \overline{t}]$ and submits a bid of $B(s)$. There is no need for i to consider bids outside of the range $[B(\underline{t}), B(\overline{t})]$; doing so can never help i and may hurt her.[7] In a symmetric equilibrium i chooses to select the bid $B(t)$ using B and her own type t. In the appendix we confirm that all bidders in an absolute auction prefer to use B rather than announce "no bid."

In a charity auction with a positive reserve and/or fees some bidders may prefer to announce "no bid" rather than a positive tender. We analyze symmetric equilibria in which all bidders with types above a threshold, $\widehat{t}$, submit bids using the increasing bidding function $B(\cdot|r, \varphi) : [\widehat{t}, \overline{t}] \to [r, \infty)$, while bidders with types below $\widehat{t}$ refrain from bidding. If r and φ are low enough, then in equilibrium all bidders submit bids and we interpret $\widehat{t}$ as equivalent to $\underline{t}$. We write the bidding function $B(\cdot|r, \varphi)$ as conditional on r and φ but not c because the level of the continuation fee does not affect a bidder's bid, given that the bidder is active. Thus the properties of $B(\cdot|r, \varphi)$ derived in this section apply to both the strong and weak auctioneer cases. We relegate to the appendix proofs that it is optimal for each bidder above and below the threshold type to follow the equilibrium strategies described in this section.

Finally, we note that without restrictions on r and φ, $B(\cdot|r, \varphi)$ may not be strictly increasing for all auctions with positive reserve prices.[8] For simplicity, we focus on values of r and φ that

[6] Alternatively, we could formulate i's problem as one of choosing an arbitrary bid, b_i, while the other bidders use the function B. This would not affect our results on equilibrium bidding strategies or revenue rankings. We adopt the signalling approach for notational ease.

[7] In all three auction formats a bid above $B(\overline{t})$ cannot improve i's probability of victory (which is one at $B(\overline{t})$) but may result in an unnecessarily high payment. A bid of $B(\underline{t})$ or below implies that i will lose the auction with certainty. The reasons why bids below $B(\underline{t})$ (when feasible) do not make i better off depend on the rules of each auction; this should be apparent to the reader after the equilibrium bid functions are derived in Section 3.

[8] For some first- and second-price auctions we conjecture that $B(\cdot|r, \varphi)$ is constant for an interval of values that

yield separating equilibria for bidders with types above $\widehat{t}$. We show below that this restriction is innocuous in the revenue-maximization problem of a strong auctioneer, and weak auctioneers prefer $r = 0$ (and a suitably chosen $\varphi > 0$) to any auction with a separating equilibrium and $r > 0$.

3.1 First-price auctions

In an absolute auction, all n potential bidders submit bids in equilibrium. In a first-price auction bidder i's return depends on her type, her bid, and the highest bid made by the other bidders. The distribution function of the highest of the $(n-1)$ other bidders' valuations is $F_1^{n-1}(x) = F(x)^{n-1}$. If B_1 is the bid function used in a first-price auction, and x is the highest valuation among the bidders other than i, then the highest bid made by the others is $B_1(x)$. Suppose bidder i imitates a bidder with type s and bids $B_1(s)$. We divide the support of x into two regions, above and below s. If $s > x$ bidder i pays $B_1(s)$ to the charity and receives the auctioned item (worth t to i) and a return of $\theta B_1(s)$ from auction revenue. When $s < x$ bidder i receives λ times the payment made by the auction winner. In total, the expected return to i from imitating a bidder of type s is

$$\pi(s|t) = \int_{\underline{t}}^{s} [t - (1-\theta)B_1(s)]dF_1^{n-1}(x) + \lambda \int_{s}^{\overline{t}} B_1(x)dF_1^{n-1}(x). \tag{1}$$

To further consider the effect of θ and λ on bidding, divide (1) into the sum of a bidder's expected return from a standard (non-charity) auction and a payoff function, Φ_1, that includes all charity-related effects:

$$\begin{aligned} \pi(s|t) &= F(s)^{n-1}[t - B_1(s)] + \Phi_1(s), \\ \text{with } \Phi_1(s) &= \theta F(s)^{n-1}B_1(s) + \lambda \int_{s}^{\overline{t}} B_1(x)dF_1^{n-1}(x). \end{aligned}$$

If bidder i selects a value of s while ignoring the terms collected in Φ_1, she faces the standard trade-off between increasing her chance of winning the auction and increasing her expected payment. The terms in Φ_1 provide i with an additional incentive to increase her choice of s, given that other bidders use the fixed bid function B_1. When the bid function is strictly increasing and $\theta \geq \lambda$, Φ_1 is increasing in a bidder's choice of s:

$$\Phi_1'(s) = \theta B_1'(s)F(s)^{n-1} + (\theta - \lambda)B_1(s)\frac{dF_1^{n-1}(x)}{ds}.$$

If a bidder increases her choice of s by a small amount, she directly benefits by θ times the increase in her expected payment. Additionally, the increased chance of i's own payment upon winning the

begins at $\widehat{t}$ and continues to a second threshold, $\widetilde{t}$. Above $\widetilde{t}$, the bid function is strictly increasing.

auction is at least as valuable as the reduced probability of another bidder's payment because of the warm glow $(\theta - \lambda)$ from transfers to the charity. Together, these points suggest that bidding in a first-price charity auction is more aggressive than in an auction with $\Phi_1 = 0$.[9]

In a symmetric equilibrium, it is optimal for i to select $s = t$ (her own type). Regardless of auction format, the first-order condition for an incentive-compatible selection of s is

$$\left.\frac{\partial \pi(s|t)}{\partial s}\right|_{s=t} = 0. \tag{2}$$

When (2) is applied to the expected return given in (1) for a first-price charity auction, we obtain

$$[t - (1 - \theta + \lambda)B_1(t)]\frac{dF_1^{n-1}(t)}{dt} - (1 - \theta)B_1'(t)F_1^{n-1}(t) = 0. \tag{3}$$

Recall that $\Delta = \theta - \lambda$, and rearrange (3) to obtain the differential equation

$$B_1'(t) + B_1(t)\left(\frac{1-\Delta}{1-\theta}\right)\frac{\frac{d}{dt}\left[F(t)^{n-1}\right]}{F(t)^{n-1}} = \frac{t}{1-\theta}\frac{\frac{d}{dt}\left[F(t)^{n-1}\right]}{F(t)^{n-1}}. \tag{4}$$

The term $\frac{(n-1)(1-\Delta)}{(1-\theta)}$ would appear frequently in the discussion below, so we replace it with α for simplicity. To solve the differential equation (4) we multiply each side of the expression by the integrating factor $F(t)^{\alpha}$ to obtain

$$\frac{d}{dt}[B_1(t)F(t)^{\alpha}] = \frac{t}{1-\Delta}\frac{d}{dt}[F(t)^{\alpha}].$$

Integrating from $\underline{t}$ to t (along with the boundary condition $B_1(\underline{t})F(\underline{t})^{\alpha} = 0$) yields the bid function for a first-price charity auction:

$$B_1(t) = \frac{1}{1-\Delta}\left[t - \int_{\underline{t}}^{t}\left(\frac{F(x)}{F(t)}\right)^{\alpha} dx\right]. \tag{5}$$

The derivation above establishes that if there is an equilibrium bidding function of the kind specified, it is necessarily $B_1(t)$, given by equation (5). We establish the converse in the appendix, that this function (and the other bid functions derived below) satisfies conditions sufficient for equilibrium. The derivation also establishes that B_1 is the unique equilibrium among all possible symmetric, increasing, and differentiable bidding rules for bidders with valuations in $(\underline{t}, \overline{t}]$. The strategy for a bidder with the lowest valuation $\underline{t}$ is indeterminate – any bid between 0 and $\frac{t}{1-\Delta}$ is optimal. A similar indeterminacy at the boundary points of t's support can occur in the second-price charity

[9] This does not prove that bidding in a charity auction is more aggressive than in a standard auction. We have demonstrated that the charity component of this auction leads one bidder to increase her bid, holding fixed the bidding strategies of the other auction participants. We still must verify that bidding is more aggressive in equilibrium.

auction too. However, these issues do not affect the expected revenue of either auction because the indeterminacy is restricted to subsets of the support with zero probability measure.

When there is only an egotistical return from charitable payments ($\theta > 0$ and $\lambda = 0$), bids in a charity auction are simply those that would be submitted in a standard auction ($\theta = \lambda = 0$), but scaled up by the constant $(1-\theta)^{-1}$. The other borderline case is that of purely altruistic behavior ($\theta = \lambda > 0$). In this situation, the bid function $B_1(t)$ leads to higher bids than in the non-charity case because bidder i behaves as if she has more than $(n-1)$ competitors for the auctioned item.[10] When the warm glow increases through a change to θ (holding λ fixed), bids increase as well. Finally, we note that the bid function $B_1(t)$ is decreasing in λ, as an increase in the attractiveness of other bidders' payments crowds out one's own incentive to bid aggressively.[11]

3.1.1 Reserve prices, bidding fees, and bidder utility

Now suppose that the auctioneer sets a bidding fee of φ to be paid by all active bidders, a continuation fee of c that a strong auctioneer can extract from both active and inactive bidders, and a reserve price of r. We consider reserve prices $r \geq 0$ that generate a separating equilibrium among active bidders; the restrictions sufficient to ensure this are given below.

As above, we specify the (potential) bidder's problem as one of selecting the optimal signal, s, to report to the auctioneer. In equilibrium, when a bidder has a type less than $\widehat{t}$, the bidder announces "no bid". If $\varphi = 0$ and r is sufficiently small, $\widehat{t} = \underline{t}$. However, whenever $\varphi > 0$ some bidders opt out of the auction, and there exists a $\widehat{t} > \underline{t}$. If the auctioneer is strong, all bidders pay the fee c to avoid dissolution of the auction. In order for this to occur, c must not be so large that the non-participants prefer the status quo (zero utility) to paying c and subsequently observing the auction unfold. In an auction with a weak auctioneer, bidders with types below $\widehat{t}$ make no payments to the auctioneer. For either type of auctioneer, bidders with types between $\widehat{t}$ and $\overline{t}$ pay φ and use the same strictly increasing bid function $B_1(\cdot|r,\varphi) : [\widehat{t}, \overline{t}] \to [r, \infty)$.

We begin with the signalling problem of a bidder with type $t > \widehat{t}$, who selects a value of $s \in [\widehat{t}, \overline{t}]$. The bidder's expected return from her bid, conditional on bidding and all other bidders following

[10] If we solve $m - 1 = \frac{n-1}{1-\theta}$ for m, we can say that bidders behave as if there are $m = \frac{n-\theta}{1-\theta}$ potential bidders in a standard auction rather than n. While m is not guaranteed to be an integer, if a bidder in a standard auction views her probability of winning as F^{m-1} rather than F^{n-1}, the equilibrium bidding function is the same as B_1 for $\theta = \lambda > 0$.

[11] To see this, verify that $\frac{\partial B}{\partial \lambda} < 0$ when $t = \underline{t}$, and that $\frac{\partial^2 B}{\partial \lambda \partial t} < 0$.

their posited equilibrium strategies, is

$$\pi(s|t) = \int_{\underline{t}}^{s} [t-(1-\theta)B_1(s)]dF_1^{n-1}(x) + \lambda \int_{s}^{\overline{t}} B_1(x)dF_1^{n-1}(x)$$
$$-\left\{1-\theta-\lambda(n-1)[1-F(\widehat{t})]\right\}\varphi - [1-\theta-\lambda(n-1)]c.$$

In the weak auctioneer case $c = 0$. The first-order condition of bidder i's signalling problem is the same as in the absolute auction. However, inspection of the differential equation (3) reveals that the bidding function is not increasing without a restriction on r and/or φ. The slope of the bidding function is

$$B_1'(t|r,\varphi) = [t-(1-\Delta)B_1(t|r,\varphi)]\frac{1}{(1-\theta)F(t)^{n-1}}\frac{dF_1^{n-1}(t)}{dt}.$$

We require r and φ to jointly yield a value of $\widehat{t}$ such that $\widehat{t} \geq (1-\Delta)r$, which ensures that $t > (1-\Delta)B_1(t|r,\varphi)$ and the bid function is increasing. Under this assumption, the equilibrium bidding function $B_1(t|r,\varphi)$ is obtained by integrating the differential equation (4) from $\widehat{t}$ to t. If $\varphi = 0$ and $r \leq \frac{\underline{t}}{1-\Delta}$, the same boundary condition applies as when there are no reserve or fees, and $B_1(t|r,\varphi) = B_1(t|0,0)$. However, if these conditions on φ and r are not met, $B_1(\widehat{t}|r,\varphi) = r$ is the boundary condition which pins down the bidding function, and

$$B_1(t|r,\varphi) = \frac{1}{1-\Delta}\left[t - \int_{\widehat{t}}^{t}\left(\frac{F(x)}{F(t)}\right)^{\alpha} dx\right] - \left(\frac{\widehat{t}}{1-\Delta} - r\right)\frac{F(\widehat{t})^{\alpha}}{F(t)^{\alpha}}.$$

We now characterize how the value of $\widehat{t}$ is determined and complete our discussion of the restrictions on r and φ. When all bidders adhere to the proposed equilibrium strategies for the case of a strong auctioneer, the expected utility to a potential bidder who opts out of the bidding (but pays c) is

$$U_1^{out}(\widehat{t}) = \lambda \int_{\widehat{t}}^{\overline{t}} B_1(x|r,\varphi)dF_1^{n-1}(x) + \lambda(n-1)[1-F(\widehat{t})]\varphi - [1-\theta-\lambda(n-1)]c.$$

If the auctioneer is weak, $c = 0$. When a bidder of type $\widehat{t}$ participates in the auction, her expected utility is

$$U_1^{in}(\widehat{t}) = F(\widehat{t})^{n-1}[\widehat{t}-(1-\theta)r] - (1-\theta)\varphi + U_1^{out}(\widehat{t}).$$

All bidders submit bids ($\widehat{t} = \underline{t}$) when $r \leq \frac{\underline{t}}{1-\Delta}$ and $\varphi = 0$, which ensures $U_1^{in}(\underline{t}) \geq U_1^{out}(\underline{t})$. There exists a binding threshold type $\widehat{t} > \underline{t}$ when $U_1^{out}(\widehat{t}) = U_1^{in}(\widehat{t})$ for a value $\widehat{t} \in (\underline{t}, \overline{t})$. The condition that uniquely determines $\widehat{t}$ is

$$F(\widehat{t})^{n-1}\widehat{t} = (1-\theta)[F(\widehat{t})^{n-1}r + \varphi]. \qquad (6)$$

Combining this condition with the restriction $\widehat{t} \geq (1-\Delta)r$, we see that φ and r must jointly yield a value $\widehat{t}$ such that $\varphi \geq \frac{\lambda}{1-\theta} r F(\widehat{t})^{n-1}$ for the bid function to be strictly increasing.

The auctioneer also (endogenously) places upper limits on values of r, φ, and c so that some bidders prefer to participate in the auction. Strong and weak auctioneers must choose r and φ to satisfy $\bar{t} > (1-\theta)(r+\varphi)$ or else no bidder would ever submit a bid. Additionally, the strong auctioneer must select a value for c which guarantees, for given r and φ, that payment of c is preferred to utility of zero. That is, c must satisfy

$$\frac{\lambda}{\Omega}\left\{\int_{\widehat{t}}^{\bar{t}} B_1(x|r,\varphi)dF_1^{n-1}(x) + (n-1)[1-F(\widehat{t})]\varphi\right\} \geq c. \tag{7}$$

Similar restrictions also apply to the second-price and all-pay auctions, and we assume that they are satisfied in the analysis below. In Section 4 we return to the optimal selection of r, φ, and c.

3.2 Second-price auctions

As above, we begin with the case of an auctioneer who does not set fees or a minimum bid. In a second-price absolute auction the payoff to bidder i depends on her type, her bid, and the highest and second-highest bids submitted by the other participants in the auction. The distribution function of the second-highest type of the $(n-1)$ other bidders is $F_2^{n-1}(x) = \{(n-1)F(x)^{n-2}[1-F(x)] + F(x)^{n-1}\}$. We assume that all bidders other than i use the function B_2 to map their (implicitly announced) tastes for the auctioned item into bids. The possible outcomes can be divided into three cases. First, i wins the auction because she signals a type, s, that is greater than the highest (and second-highest) type of the $(n-1)$ other bidders. This case results in i paying the highest bid submitted by the other auction participants. In return, i receives the prize (worth t) and a benefit of θ for each dollar she pays. Second, with probability $(n-1)F(s)^{n-2}[1-F(s)]$ bidder i submits the second-highest bid. Since the winner pays the second-highest bid, i receives a return of $\lambda B_2(s)$. Third, the type selected by i is smaller than the first- and second-highest types of the other $(n-1)$ bidders. The winner pays the second-highest bid and i's return is λ for each dollar of auction revenue. Combining these three cases, the expected return to bidder i with type t from imitating a bidder of type s is

$$\begin{aligned}\pi(s|t) \quad &= \quad \int_{\underline{t}}^{s}[t-(1-\theta)B_2(x)]dF_1^{n-1}(x) + \lambda(n-1)F(s)^{n-2}[1-F(s)]B_2(s) \\ &\quad +\lambda\int_{s}^{\bar{t}} B_2(x)dF_2^{n-1}(x).\end{aligned}$$

Again, we split i's expected return from the auction into a non-charity component and $\Phi_2(s)$, the bidder's charity-related surplus when she mimics a bidder with valuation s:

$$\begin{aligned} \pi(s|t) &= \int_{\underline{t}}^{s} [t - B_2(x)] dF_1^{n-1}(x) + \Phi_2(s), \\ \text{with } \Phi_2(s) &= \theta \int_{\underline{t}}^{s} B_2(x) dF_1^{n-1}(x) + \lambda(n-1)F(s)^{n-2}[1-F(s)]B_2(s) \\ &\quad + \lambda \int_{s}^{\bar{t}} B_2(x) dF_2^{n-1}(x). \end{aligned}$$

The term Φ_2 is increasing in s, suggesting that participants in a charity auction bid more aggressively than in a standard auction:

$$\Phi_2'(s) = \lambda(n-1)F(s)^{n-2}[1-F(s)]B_2'(s) + (\theta - \lambda)B_2(s)\frac{dF_1^{n-1}(s)}{ds}.$$

The first term in Φ_2' is the increase in expected surplus from placing second with a slightly higher bid. The second term accounts for the increased probability that i wins the auction; this term is nonnegative because i's charitable sentiment for her own payments is at least as strong as the return from other bidders' payments.

The first-order condition (2) yields

$$[t - (1-\Delta)B_2(t)]\frac{dF_1^{n-1}(t)}{dt} + \lambda(n-1)F(t)^{n-2}[1-F(t)]B_2'(t) = 0. \tag{8}$$

Simplifying (8) yields the differential equation

$$B_2'(t) - B_2(t)\left(\frac{1-\Delta}{\lambda}\right)\left(\frac{f(t)}{1-F(t)}\right) = -\frac{t}{\lambda}\left(\frac{f(t)}{1-F(t)}\right) \tag{9}$$

Let β denote $\frac{1-\Delta}{\lambda}$ and multiply each side of (9) by the integrating factor $[1-F(t)]^{\beta}$ to obtain

$$\frac{d}{dt}\left\{B_2(t)[1-F(t)]^{\beta}\right\} = \left(\frac{t}{1-\Delta}\right)\frac{d}{dt}\{[1-F(t)]^{\beta}\}.$$

We integrate both sides of this expression from t to $\bar{t}$ and use the boundary condition $B_2(\bar{t})[1 - F(\bar{t})]^{\beta} = 0$ to obtain the bid function for a second-price charity auction:

$$B_2(t) = \left(\frac{1}{1-\Delta}\right)\left\{t + \int_t^{\bar{t}} \left(\frac{1-F(x)}{1-F(t)}\right)^{\beta} dx\right\} \tag{10}$$

As in a standard auction, the bid function for a second-price charity auction is independent of the number of bidders, n. Also notice that when θ and λ are zero B_2 simplifies to t, the well-known bidding rule for a second-price independent private values auction. When $\theta > 0$ and $\lambda = 0$ (purely egotistical returns from charitable giving), θ affects B_2 as a subsidy would in a standard auction. For any fixed value of λ, bids increase with θ because of the warm glow from a bidders' own payments to the charity. As in a standard auction, $B_2(t) > B_1(t)$ for all t.

The derivation also establishes that the symmetric increasing bidding equilibrium is unique for all bidders whose types lie in $(\underline{t}, \overline{t})$. If the number of bidders exceeds 2 then the bid for the lowest type is again indeterminate – any bid between 0 and $B_2(\underline{t})$ is optimal for type $\underline{t}$. Similarly, a bidder with the valuation $\overline{t}$ has an indeterminate bid, which can be any value that is at least as great as $B_2(\overline{t}) = \overline{t}/(1-\Delta)$. For simplicity we break the indeterminacy at the endpoints by assuming that the bid function is continuous on its support, but nothing essential relies on this assumption because expected revenue is unaffected by bidding behavior on a set of probability zero.

When there is an increase in λ (the utility from other bidders' payments), a crowding-out effect dampens bidding in a first-price charity auction. We do not obtain a similar result for second-price auctions because of the more complex relationship among bidders' own bids and others' payments. Bidders with valuations near $\overline{t}$ submit bids that approach $\overline{t}/(1-\Delta)$. The role of λ in Δ implies that bids are decreasing in λ for these bidders. However, bidders with low t may increase their bids with λ. This effect for low-t bidders has intuitive appeal, since these bidders are more likely to lose the auction and determine the payment of the winner rather than win the auction themselves. But this relationship does not hold for all combinations of parameter values, as it is possible that the weakened incentives for bidders with high t may effectively dampen the bids of low-t bidders in equilibrium.[12]

3.2.1 The relationship to open auctions

Second-price sealed-bid auctions are often studied because they share several characteristics with the more common open, ascending-bid (English) auction. With private values, bidding incentives in the two auction formats are very similar in the non-charity case, and expected revenues from the two auction formats are identical. We find that this parallel also applies to charity auctions. This relationship between the auctions is studied through the button auction, a stylized version of the English auction (see Milgrom and Weber (1982)).[13] We confine our analysis to the absolute

[12] For example, suppose that the valuations are distributed uniformly on $[0,1]$. Then the equilibrium bid function is $B_2(t) = (1+\beta t)/[(1-\Delta)(1+\beta)]$. Fix $\theta = 0.5$ and consider the bids of a type-0 bidder. When $\lambda = 0$, $B_2(0) = 0$; when $\lambda = 0.2$, $B_2(0) = 0.317$; and when $\lambda = 0.4$, $B_2(0) = 0.234$. Thus, the bidding function is not monotone in λ at the bottom of the type distribution.

[13] In the non-charity case the equivalence of the second-price and button auction is easy to show because the equilibrium strategy is weakly dominant. This dominance does not extend to the non-charity case.

auction, in which the auctioneer sets no fees or reserve price.

The rules of a button auction are as follows. Starting at zero, the auctioneer displays a continuously increasing price for the object being sold. Each bidder has a personal button: a depressed button means that the bidder is willing to buy the object at its current price. Once a bidder releases her button she exits the auction and cannot return. The auction is over when only one bidder is pressing her button; this bidder wins the object and pays the auctioneer the price that was posted when the second-last bidder exited. A tie occurs when, with more than one bidder left, all these bidders exit at the same price. We resolve such ties by giving those who tied an equal chance of receiving the prize and paying the price. However, because the price is increasing continuously, it is impossible for one bidder to observe another's exit and then also exit (in response) at precisely the same price. Ties occur with zero probability in equilibrium, since the types are drawn independently from a continuous distribution and bidders follow the same strictly increasing bid function.

We consider two different versions of the button auction. In the first, exit is observable: each bidder who is active at price p has observed the exit prices of all bidders who exited at prices q for $q < p$. In the second version, exit is unobservable: an active bidder at price p knows only that there is at least one other bidder who has not exited at any price below p (or else the auction would have ended before reaching p). Conditional on an equilibrium bidding rule that is strictly increasing in t, as the auction proceeds the bidders are able to infer that the types of their remaining rivals lie above some threshold $x \in (\underline{t}, \overline{t})$ and are drawn from the truncated distribution $F(s|s \geq x) = \frac{F(s)-F(x)}{1-F(x)}$ with $s \in [x, \overline{t}]$. We demonstrate here that the equilibrium bid function for a second-price sealed-bid auction, B_2, is also an equilibrium bidding (*i.e.*, exit) strategy in a button auction regardless of whether bidders observe the exit of their rivals. This result relies on two key properties of the function B_2, stated in the following lemma.

Lemma 1: *The symmetric equilibrium bid function in the second-price charity auction, B_2, is invariant with respect to: 1) a change in the number of bidders n, or 2) a replacement of the type distribution $F(s)$ by the truncated distribution $F(s|s \geq x)$.*

This lemma is crucial for proving that B_2 applies to the button auction since it shows that the strategy for a bidder of type t of waiting until $B_2(t)$ remains optimal despite any information learned during the auction. As the price rises, bidders will exit and each remaining bidder can infer that her active rivals' types are all above some value $x \in (\underline{t}, \overline{t})$. However, this information has no effect on bidding behavior.[14]

Proposition 1. *The following strategies generate a Perfect Bayesian Equilibrium in a button auction: Each bidder of type t who has not yet exited will exit at price p if and only if $p \geq B_2(t)$. The equilibrium outcome of the button auction in terms of allocations and payments is identical to that in the second-price sealed-bid auction. Moreover, these results hold both if: 1) bidders observe when rivals exit in the button auction, or 2) exit is not observed.*

3.2.2 Reserve prices, bidding fees, and bidder utility

Now consider the bidding strategies when the auctioneer sets a reserve price of r and fees of φ and c. As above, we focus on (r, φ) pairs that yield a separating equilibrium among active bidders. In the equilibrium, there may exist a threshold type $\widehat{t} \in (\underline{t}, \overline{t})$ such that all bidders with $t < \widehat{t}$ choose to announce "no bid," while bidders with $t > \widehat{t}$ use the increasing bidding function $B_2(\cdot|r, \varphi) : [\widehat{t}, \overline{t}] \to [r, \infty)$. However, for some pairs (r, φ) with r and φ positive, all bidders will bid using $B_2(\cdot|r, \varphi)$, so $\widehat{t} = \underline{t}$. Following our discussion of the increasing function B_2 and the conditions that determine $\widehat{t}$, we state a sufficient condition that ensures separation among active bidders.

We begin with the bidding problem of a bidder with $t > \widehat{t}$, where $\widehat{t}$ may equal $\underline{t}$. When bidder i with valuation t imitates a bidder of type with $s \in [\widehat{t}, \overline{t}]$, her expected return from bidding is

$$\begin{aligned}
\pi(s|t) \quad = \quad & \int_{\widehat{t}}^{s} [t - (1-\theta)B_2(x|r,\varphi)]dF_1^{n-1}(x)^{n-1} + F(\widehat{t})^{n-1}[t - (1-\theta)r] \\
& + \lambda(n-1)F(s)^{n-2}[1 - F(s)]B_2(s|r,\varphi) + \lambda \int_{s}^{\overline{t}} B_2(x|r,\varphi)dF_2^{n-1}(x) \\
& - \left\{1 - \theta - \lambda(n-1)[1 - F(\widehat{t})]\right\} \varphi - [1 - \theta - \lambda(n-1)]c,
\end{aligned}$$

with $c = 0$ when the auctioneer is weak. If i is the only active bidder, she pays the reserve price of r. This occurs with probability $F(\widehat{t})^{n-1}$. Despite this detail in an auction with a reserve, the optimal selection of a type $s \in [\widehat{t}, \overline{t}]$ is similar to an absolute auction. Provided that r is not too large, the same differential equation and boundary condition apply here as above. Thus, for $t \geq \widehat{t}$ the bidding function $B_2(t|r, \varphi)$ is equal to $B_2(t|0, 0)$, or simply $B_2(t)$ from the absolute auction.

Next, we consider how the value of $\widehat{t}$ is selected so that all bidders are willing to follow the

[14] The opportunity to observe when rivals exit allows other symmetric Perfect Bayesian Equilibria even in the standard (*i.e.* non-charity) case, but only the inframarginal bidding is affected, not the winning bid or revenue. See Bikhchandani, Haile, and Riley (2002).

proposed equilibrium. In an auction with a strong auctioneer, a bidder with type $\widehat{t}$ receives utility

$$\begin{aligned} U_2^{out}(\widehat{t}) &= \lambda(n-1)F(\widehat{t})^{n-2}[1-F(\widehat{t})]r + \lambda \int_{\widehat{t}}^{\bar{t}} B_2(x)dF_2^{n-1}(x) \\ &\quad +\lambda(n-1)[1-F(\widehat{t})]\varphi - [1-\theta-\lambda(n-1)]c. \end{aligned}$$

when she announces "no bid" and pays c, conditional on all other bidders paying c also. If the bidder with type $\widehat{t}$ chooses to submit a positive bid of $B_2(\widehat{t})$, her expected utility is

$$U_2^{in}(\widehat{t}) = F(\widehat{t})^{n-1}[\widehat{t} - (1-\theta)r] - (1-\theta)\varphi + \lambda(n-1)F(\widehat{t})^{n-2}[1-F(\widehat{t})][B_2(\widehat{t}) - r] + U_2^{out}(\widehat{t}).$$

Of course, if the strong auctioneer were to choose a value of c so that $U_2^{out}(\widehat{t}) < 0$, all bidders would opt out of the auction and receive zero utility. The weak auctioneer case is different only in that c must equal zero in $U_2^{out}(\widehat{t})$.

All bidders submit bids if the utility to the bidder with type $\underline{t}$ from participating is greater than from opting out. That is, $\widehat{t} = \underline{t}$ if

$$U_2^{in}(\underline{t}) - U_2^{out}(\underline{t}) = \lambda(n-1)F(\underline{t})^{n-2}[B_2(\underline{t}) - r] - (1-\theta)\varphi > 0.$$

If the auctioneer sets $\varphi = 0$ and $B_2(\underline{t}) > r$, the reserve price has no effect on participation decisions or auction revenue. Additionally, if $n = 2$ and $B_2(\underline{t}) > r$ a bidder with type $\underline{t}$ is willing to pay a positive bidding fee φ in order to determine the payment of the winning bidder.

When a nontrivial threshold, $\widehat{t} \geq \underline{t}$, exists, it is characterized by indifference between using B_2 and announcing "no bid." Setting $U_2^{out}(\widehat{t}) = U_2^{in}(\widehat{t})$ yields

$$F(\widehat{t})^{n-1}[\widehat{t} - (1-\theta)r] + \lambda(n-1)F(\widehat{t})^{n-2}[1-F(\widehat{t})][B_2(\widehat{t}) - r] = (1-\theta)\varphi \tag{11}$$

We show in the appendix that this expression uniquely determines the value of $\widehat{t}$, conditional on the use of the bidding function B_2 and $B_2(\widehat{t}) \geq r$. However, it is possible to have a pair of values (r, φ) such that (11) is satisfied while $B_2(\widehat{t}) < r$, which is not permitted under the rules of the auction.[15] To ensure that this does not occur, following assumption is sufficient.

Assumption: *Hold fixed the bidding rule B_2 in equation (10) and consider a pair (r, φ). If it is the case that (11) holds where $B_2(\widehat{t}) < r$, then the pair (r, φ) is not permitted in the auction.*

[15] For example, suppose $t \sim U[0,1]$, $\theta = \lambda = 0.4$, and $n = 2$. In this case, $B_2(t) = \frac{1+2.5t}{3.5}$. If the auctioneer sets $r = 0.8$ and $\varphi = 0$, the indifference condition in equation (11) is satisfied at $\widehat{t} = 0.52$. However, at this $\widehat{t}$, $B_2(\widehat{t}) = 0.65$, which is not permitted by the rules of the auction. The minimum bidder type who can use B_2 and select a permitted bid is $\widetilde{t} = 0.72$. But if bidders with types above 0.72 use B_2 while others are inactive, a bidder with a type just below $\widetilde{t}$ can imitate the bid of a type-$\widetilde{t}$ person and experience a strict increase in utility.

While this assumption clearly places a restriction on how the auctioneer can use r and φ to raise revenue, the assumption does not restrict his ability to choose which types of bidders participate in the auction. It is straightforward to show that the auctioneer can satisfy $B_2(\widehat{t}) \geq r$ and select any $\widehat{t} \in [\underline{t}, \overline{t})$ he prefers with appropriate values of r and φ.

3.3 All-pay auctions

We begin, again, by considering an absolute auction. In an all-pay auction the highest bidder wins the object but all auction participants must pay their bids. Bidder i's return depends on her type, her bid, and the bids of the $(n-1)$ other bidders. Let x represent the valuation of the bidder other than i with the highest type, and assume that all bidders other than i use the function B_A to map valuations into bids. When i mimics an individual with type s she bids $B_A(s)$, and the highest bid submitted by the other auction participants is $B_A(x)$. If $s > x$ bidder i receives the prize, which she values at t, and pays $B_A(s)$. Bidder i loses the auction when $s < x$, but she still must pay her bid of $B_A(s)$. The two possible auction outcomes are combined to yield i's expected return from an all-pay charity auction:

$$\pi(s|t) = \int_{\underline{t}}^{s} t dF_1^{n-1}(x) - (1-\theta)B_A(s) + \lambda(n-1)\int_{\underline{t}}^{\overline{t}} B_A(x)dF(x). \tag{12}$$

The all-pay structure of this auction is evident in that i's bid of $B(s)$ is paid regardless of the identity of the auction winner. An important feature of (12) is that i's choice of s does not affect her benefit from the payments made by the $(n-1)$ other bidders. When we separate $\pi(s|t)$ into non-charity and charity components, we obtain

$$\pi(s|t) = tF(s)^{n-1} - B_A(s) + \Phi_A(s),$$
$$\text{with } \Phi_A(s) = \theta B_A(s) + \lambda(n-1)\int_{\underline{t}}^{\overline{t}} B_A(x)dF(x).$$

Bidder i's surplus from the auctioneer's revenue increases in her own reported type, s. Additionally, i's charity-related incremental surplus from increasing s is just $\Phi'_A = \theta B'_A(s)$, which is independent of λ and other bidders' payments.

As in single-price auctions, incentive compatibility is captured by condition (2). Differentiating (12), restricting $s = t$, and setting the result to zero yields

$$\begin{aligned} t\frac{dF_1^{n-1}(t)}{dt} - (1-\theta)B'_A(t) &= 0, \\ \text{or } \frac{d}{dt}\left[B_A(t)\right] &= \frac{t}{1-\theta}\frac{dF(t)^{n-1}}{dt}. \end{aligned} \tag{13}$$

We integrate (13) from $\underline{t}$ to t, employ the boundary condition $B_A(\underline{t}) = 0$,[16] and find that the bid function for an all-pay charity auction is

$$B_A(t) = \frac{1}{1-\theta}\left[tF(t)^{n-1} - \int_{\underline{t}}^{t} F(x)^{n-1}dx\right]. \tag{14}$$

The function (14) differs from bid functions in non-charity all-pay auctions only in the factor $\frac{1}{1-\theta}$. In first- and second-price charity auctions bidding is affected by the value of λ, but this is clearly not the case in an all-pay auction. Although a bidder benefits from an increase in λ because her utility from others' payments rises, a change in λ does not affect bidding incentives.

3.3.1 Reserve prices, bidding fees, and bidder utility

Suppose that the auctioneer can set a minimum bid of r, a fee of φ to be paid by active bidders, and a fee of c that a strong auctioneer collects from all bidders. As long as r or φ is positive there exists a $\widehat{t} > \underline{t}$ such that in equilibrium bidders with types below $\widehat{t}$ announce "no bid." If these bidders face a strong auctioneer they pay c; with a weak auctioneer inactive bidders do not pay this fee. Bidders with types at or above $\widehat{t}$ pay φ and use the same increasing function $B_A(\cdot|r,\varphi) : [\widehat{t},\overline{t}] \to [r,\infty)$ whether the auctioneer is weak or strong. When i's type is above $\widehat{t}$, she prefers to select a type, s, in $[\widehat{t},\overline{t}]$ rather than announcing "no bid." Player i's expected benefit from her bid, conditional on bidding, when she announces an $s \in [\widehat{t},\overline{t}]$ is

$$\begin{aligned}\pi(s|t) &= \int_{\underline{t}}^{s} tdF_1^{n-1}(x) - (1-\theta)B_A(s|r,\varphi) + \lambda(n-1)\int_{\widehat{t}}^{\overline{t}} B_A(x|r,\varphi)dF(x) \\ &\quad - \left\{1-\theta-\lambda(n-1)[1-F(\widehat{t})]\right\}\varphi - [1-\theta-\lambda(n-1)]c.\end{aligned}$$

When the auctioneer is weak, $c = 0$. The first-order condition that characterizes an optimal choice of s for bidder i is exactly as the absolute auction, and leads to the same differential equation. $B_A'(t|r,\varphi)$ is always positive, and no further assumptions are necessary to ensure a separating equilibrium. We integrate the differential equation from $\widehat{t}$ to t and apply the boundary condition $B_A(\widehat{t}|r,\varphi) = r$ to obtain the bid function

$$B_A(t|r,\varphi) = \frac{1}{1-\theta}\left\{tF(t)^{n-1} - \widehat{t}F(\widehat{t})^{n-1} - \int_{\widehat{t}}^{t} F(x)^{n-1}dx\right\} + r.$$

[16] To understand why $B_A(\underline{t}) = 0$, suppose that the equilibrium bid function has $B_A(\underline{t}) > 0$. In the posited equilibrium a bidder with the lowest possible type has no chance of winning the auction but still makes a payment to the auctioneer. This implies that for type $\underline{t}$ the first term in (12) is zero and the second is negative. The bidder's return would increase with a reduction in $B_A(\underline{t})$, therefore no bid function with $B_A(\underline{t}) > 0$ is an admissible equilibrium bidding rule.

We now characterize the type $\widehat{t}$, which divides active bidders from those who announce "no bid." If the auctioneer is strong and can collect payment of c from all bidders, the utility to a bidder of type $\widehat{t}$ from announcing "no bid" is

$$U_A^{out}(\widehat{t}) = \lambda(n-1)\int_{\widehat{t}}^{\overline{t}} B_A(x)dF(x) + \lambda(n-1)[1-F(\widehat{t})]\varphi - [1-\theta-\lambda(n-1)]c.$$

The same utility accrues to all potential bidders with valuations $t < \widehat{t}$. The weak auctioneer case corresponds to $c = 0$. If the bidder with value $\widehat{t}$ submits a bid of r, then her expected utility is

$$U_A^{in}(\widehat{t}) = F(\widehat{t})^{n-1}\widehat{t} - (1-\theta)[r+\varphi] + U_A^{out}(\widehat{t}).$$

The threshold $\widehat{t}$ is the unique value that equates $U_A^{out}(\widehat{t})$ and $U_A^{in}(\widehat{t})$, *i.e.*, the $\widehat{t}$ that solves

$$F(\widehat{t})^{n-1}\widehat{t} = (1-\theta)(r+\varphi). \tag{15}$$

In the appendix we demonstrate that bidders of all types prefer to follow the equilibrium strategies described above.

4 Revenue Results

4.1 Notation

Consider the perspective of an auctioneer who must design an auction without knowledge of bidders' realized valuations. Each auction's expected revenue depends on *ex ante* (to the auctioneer) values of bidders' expected payments, which can be expressed as functions of bidders' utility. We assume that the bidding equilibrium in each auction is symmetric, and therefore all bidders in an auction share the same expected utility conditional on type, $U_g(t)$. Let Y_g represent the *ex ante* expected utility of a bidder in an auction of format g, so $Y_g = E_t[U_g(t)]$. Next, let $x_g(t)$ be the expected payment (conditional on type) of a bidder in a format-g auction. Then the *ex ante* expected payment of a potential bidder is $X_g = E_t[x_g(t)]$. An important implication of the *ex ante* symmetry of bidders is that the *ex ante* expected payment is identical across potential bidders. Finally, let $p_g(t)$ be the probability that a bidder of type t wins an auction of format g. Then the *ex ante* expected benefit from winning the object is denoted $W_g = E_t[p_g(t)t]$. In an auction with an increasing bid function, $p_g(t) = F(t)^{n-1}$ if the bidder submits a bid and $p_g(t) = 0$ otherwise.

For each potential bidder, the *ex ante* expected utility from the auction depends on a linear combination of her expected return from winning the valued object (W_g), the expected payments made by the bidder herself (X_g), and the expected payments of all other bidders ($(n-1)X_g$). This allows us to write *ex ante* expected utility as

$$Y_g = W_g - [1 - \theta - \lambda(n-1)]X_g. \tag{16}$$

This expression demonstrates our need for a restriction on the combination of parameter values $\Omega = [1-\theta-\lambda(n-1)]$ when the auctioneer is unrestricted in his ability to design revenue-generating mechanisms. If $\Omega < 0$, a bidder's utility (Y_g) is increasing in the expected payments that follow from the mechanism, and each bidder would be willing to participate in a mechanism that has all bidders transferring any fixed amount of money to the auctioneer.[17]

The auctioneer's expected revenue from a format-g auction, R_g, is the sum of the expected payments from all n potential bidders: $R_g = nX_g$. From equation (16),

$$\Omega R_g = nW_g - nY_g. \tag{17}$$

To facilitate revenue comparisons, we note that the envelope condition $U_g'(t) = p_g(t)$ allows us to write

$$U_g(t) = U_g(\underline{t}) + \int_{\underline{t}}^{t} p_g(s)ds. \tag{18}$$

This implies that any pair of auctions with the same p_g and $U_g(\underline{t})$ provide the same expected payoff to bidders of all types. Additionally, a pair of auctions with the same p_g offer the same *ex ante* benefit from winning the object, W_g. From (17), this payoff-equivalence across auctions for bidders implies revenue-equivalence for the auctioneer.

We substitute (18) into the definition of Y_g and switch the order of integration to obtain

$$Y_g = U_g(\underline{t}) + \int_{\underline{t}}^{\overline{t}} [1 - F(t)]p_g(t)dt.$$

When we combine this expression for Y_g with the definition of W_g, we can write (17) as

$$\Omega R_g = \int_{\underline{t}}^{\overline{t}} \left\{ t - \frac{[1 - F(t)]}{f(t)} \right\} p_g(t)nf(t)dt - nU_g(\underline{t}). \tag{19}$$

The right-hand-side of equation (19) is familiar from Myerson's optimal auction analysis. As is standard, we assume that the distribution of types is such that a bidder's virtual valuation,

$$t - \frac{[1 - F(t)]}{f(t)},$$

[17] In this case it is not even necessary for the auctioneer to award a prize to any bidder.

is increasing in t. This is sufficient to ensure that the mechanism design problem is "regular" (Krishna, 2002). There are two important differences to note between (19) and the analogous term for a standard auction. First, the expression is equal to ΩR_g and not simply R_g. This is important for revenue comparisons among auctions when we permit $\Omega < 0$, as in the absolute auction and when the auctioneer is weak. Second, a simple but important characteristic of the optimal auction is that the auctioneer chooses the mechanism so that $U_g(\underline{t}) = 0$. Similarly, in a charity auction with a strong auctioneer the continuation fee is set so that $U_g(\underline{t}) = 0$. This allow us to extend the traditional revenue equivalence result and characterization of the optimal auction to the strong auctioneer case. However, in absolute auctions or when the auctioneer is weak, the auctioneer is unable to extract all surplus from the bidder with type $\underline{t}$, so this bidder and all inactive bidders receive positive utility from the mechanism. Each auction format typically leaves the type-$\underline{t}$ bidder with a different amount of surplus, which leads to revenue differences across auctions.

A minor extension to equation (19) is useful to specify here. In an auction with a strictly increasing bidding function, $p_g(t)nf(t)$ is $\frac{dF_1^n(t)}{dt}$ for active bidders and zero otherwise. Let $\widehat{t}$ represent the lowest valuation of an active bidder, where $\widehat{t} = \underline{t}$ in an absolute auction and for sufficiently small values of r and φ. Then, given a threshold value $\widehat{t}$ and our focus on auctions with separating equilibria, we may write (19) as

$$\Omega R_g = \int_{\widehat{t}}^{\overline{t}} \left\{ t - \frac{[1 - F(t)]}{f(t)} \right\} dF_1^n(t) - nU_g(\widehat{t}). \tag{20}$$

Thus, revenue comparisons between auction formats which share a threshold value of $\widehat{t}$ depend entirely on the relationship between the auctions' values of $U_g(\widehat{t})$.

4.2 Revenue in an Absolute Auction

Suppose the auctioneer sets no reserve price or fees. Then all potential bidders will bid and expected revenue in a format-g auction is given by equation (20) with $\widehat{t} = \underline{t}$. Since bidders of all types receive utility from other bidders' payments to the auctioneer, $U_g(\underline{t})$ is neither zero nor necessarily equal across auctions. In comparing absolute auction revenue across auction formats, it is convenient to take the all-pay auction as a base case. In this auction, the expected utility of a bidder with the lowest possible type is proportional to the expected bids of the $n-1$ other bidders: $U_A(\underline{t}) = \lambda(n-1)X_A$. This attribute of $U_A(\underline{t})$ along with the identity $\Omega R_A = [1-\theta-\lambda(n-1)]nX_A$, allows us to write all-pay revenue as

$$R_A = \frac{1}{1-\theta} \int_{\underline{t}}^{\overline{t}} \left\{ t - \frac{[1-F(t)]}{f(t)} \right\} dF_1^n(t). \tag{21}$$

This is the same expected revenue expression as in a standard (non-charity) auction, except it is inflated by $(1-\theta)^{-1}$. The revenue differences among all-pay and first- and second-price auctions follow from the impossibility of such a simplification for either winner-pay format.

Proposition 2. *Revenue across absolute auction formats is equal when $\lambda = 0$. For $\lambda > 0$, all-pay expected revenue is greater than first-price, but the revenue ranking of all-pay and second-price absolute auctions depends on parameter values.*

We use the relationship $U_1(\underline{t}) > \lambda(n-1)X_1$ to establish the ranking of all-pay and first-price revenue. The term $U_1(\underline{t})$ is proportional to the expected value of the highest of $(n-1)$ bids, conditional on bidders behaving as if there are n active bidders in the auction. This expected value is always greater than $(n-1)X_1$, the sum of the *ex ante* expected payments from $(n-1)$ bidders in an n-bidder auction.[18] The case of the second-price auction is more difficult, as the ranking of $U_2(\underline{t})$ and $\lambda(n-1)X_2$ is ambiguous. We demonstrate this in the example below.

Example 1: *Suppose t is distributed uniformly on $[0,1]$. Then the bid functions in second-price and all-pay auctions are*

$$B_2(t) = \left(\frac{1}{1-\Delta}\right)\left(\frac{1+\beta t}{1+\beta}\right) \qquad \text{and} \qquad B_A(t) = \left(\frac{1}{1-\theta}\right)\left(\frac{n-1}{n}\right) t^n.$$

Expected revenue from the two auction formats are

$$R_2 = \frac{\lambda(n+1) + (1-\Delta)(n-1)}{(1-\Delta)\,(1-\Delta+\lambda)\,(n+1)} \qquad \text{and} \qquad R_A = \left(\frac{1}{1-\theta}\right)\left(\frac{n-1}{n+1}\right).$$

The difference $R_A - R_2$ is proportional to

$$D = \lambda[(n-3)(1-\theta) + 2\lambda(n-1)].$$

Clearly, if $n \geq 3$ then $D > 0$ and $R_A > R_2$. If $n = 2$ then $R_A > R_2$ only if $(2\lambda + \theta) > 1$.

In ranking the revenue of first- and second-price auctions, it is convenient to write an expression for ΩR_g that is similar to (20), but has $U_g(\overline{t})$ as a boundary condition on utility instead of $U_g(\underline{t})$.

[18] While $\lambda n X_1 > U_1(\underline{t})$ because $\int B(t) dF_1^n(t) > \int B(t) dF_1^{n-1}(t)$, it is true that $U_1(\underline{t}) > \lambda(n-1)X_1$ because $\int B(t)F(t)^{n-2} f(t)dt > \int B(t)F(t)^{n-1} f(t)dt$.

In this case, we write $\Omega R_g = k - nU_g(\bar{t})$, with k constant across auctions and very similar to the integral expression in equation (20). Thus revenue differences (scaled by Ω) depend on differences in $U_g(\bar{t})$ only. $U_g(\bar{t})$ may be written $U_g(\bar{t}) = \bar{t} - (1-\theta)x_g(\bar{t})$, so the difference in expected revenue from these auctions is

$$R_2 - R_1 = \frac{n(1-\theta)}{\Omega}\left[x_2(\bar{t}) - x_1(\bar{t})\right]. \tag{22}$$

The term Ω in equation (22) implies that the signs of $(R_2 - R_1)$ and $[x_2(\bar{t}) - x_1(\bar{t})]$ are different if $\Omega < 0$, but the revenue ranking is unaffected by the sign of Ω.

Proposition 3. *$x_2(\bar{t}) - x_1(\bar{t})$ is positive when $\Omega > 0$ and negative when $\Omega < 0$. This implies that $R_2 > R_1$ for all combinations of parameter values such that $\Omega \neq 0$. In the borderline case of $\Omega = 0$, direct examination of expected revenue expressions reveals that $R_2 > R_1$.*

In light of revenue equivalence when $\lambda = 0$, this result must be attributed to the benefit from other bidders' payments on B_1 and B_2. Given that a particular bidder will lose the auction, there is an incentive (and ability) to increase the winner's payments in a second-price auction that is not present in a first-price contest. This result on expected revenue accords with common practice in charity auctions. A fairly popular format for charity auctions of multiple objects is to sell the less valuable items in a "silent auction" while guests mingle, and the items with higher expected prices are sold in oral, ascending-bid auctions.

We conclude this section by establishing that for a sufficiently large n, the expected revenue in an all-pay absolute auction is greater than from either winner-pay absolute auction. We already found that R_1 is smaller than R_A for all n whenever θ and λ are positive. To establish the revenue ranking of R_2 and R_A, we show that R_2 converges to $\frac{\bar{t}}{1-\Delta}$ as n becomes large, while R_A converges to $\frac{\bar{t}}{1-\theta}$. Whenever $\lambda > 0$, this implies $R_A > R_2$ for a sufficiently many bidders.

Proposition 4. *For a sufficiently large number of bidders, n, the expected revenue in an absolute all-pay charity auction is greater than that of a first-price or second-price absolute auction.*

The intuition for the result is as follows. As n becomes large, the expected type of the highest and second-highest valuations both converge to $\bar{t}$. Holding fixed the bid of the bidder with the second-highest valuation, we may ask under what circumstances the highest-valuation bidder is willing to to out-bid her opponent. In an all-pay auction, the bidder is willing to submit a bid that leaves her with zero additional surplus from winning (rather than withdrawing from the auction) since her benefit from other bidders' payments is unaffected by her own bid. In the single-price auctions the bidder with the highest valuation must be permitted to retain some surplus from out-bidding her opponent and making the only payment to the auctioneer. If not, this bidder could submit a bid of zero and receive a strictly positive benefit from allowing the bidder with the second-highest valuation to win the auction. The relatively high revenue from an all-pay auction with a large number of bidders invites comparison of this mechanism with other types of contests in which many people pay a small fee in return for a chance to win a prize. Consider a lottery or raffle in which proceeds of the contest are spent on a public good, as in Morgan (2000). Because the all-pay auction allocates the object for sale efficiently, the auction will raise more revenue than a lottery when the bidders' type distribution and preferences are as specified in Section 2.[19] A bidder with a strong taste for a prize submits a higher bid in an all-pay auction than she is willing to spend on lottery tickets.

4.3 Revenue with a Strong Auctioneer

Assume that the auctioneer is strong, and can cancel the auction and return all payments if any of the n bidders fail to pay the continuation fee c. Suppose that $\Omega > 0$ (or else, as pointed out earlier, a strong auctioneer could raise unlimited revenue). In a symmetric equilibrium of an auction of any format g, $U_g(\underline{t})$ is the expected payoff to the bidder with the lowest type, $\underline{t}$. The expected payoff of all other types is as at least as high, since they can achieve at least $U_g(\underline{t})$ by mimicking $\underline{t}$'s behavior.

When the auctioneer adds a continuation fee that all bidders are willing to pay, he does not alter the bidding equilibrium. In a format-g auction with fixed r and φ, the bidder may set c_g so that $U_g(\underline{t}) = 0$. Call this level of c_g the *surplus-extracting* level. For example, the surplus-extracting value of c_1 in a first-price auction is set so that the weak inequality in equation (7) binds. By equation (18) if two auctions offer identical probabilities of winning their $U_g(t)$ functions differ by a constant. Because the imposition of the surplus-extracting continuation fee ensures that $U_g(\underline{t}) = 0$, the constant is zero, and auctions are payoff-equivalent to the bidders. By equation (17), the auctions are revenue-equivalent as well.

Proposition 5. *Given two auction formats, suppose that the equilibrium probability $p_g(t)$ of any type t winning the prize is the same in both. Then, if the strong auctioneer sets the continuation fee in each auction at the surplus-extracting level, the two auctions are revenue-equivalent and*

[19] Goeree *et al.* (2005) prove that this revenue ranking holds between auctions and lotteries.

payoff-equivalent.

Since the proposition above applies to any pair of auctions with identical, weakly increasing functions p_g, the proposition also applies to the cases we study in this paper, in which a separating equilibrium exists in the selection of bids for all active bidders.

Corollary 1 *Fix $\widehat{t} \in [\underline{t}, \overline{t})$. Consider a pair of auctions in which types $t < \widehat{t}$ do not bid (if such types exist) while types $t > \widehat{t}$ bid according to a symmetric and strictly increasing bid function. The auctions' bid functions may be distinct. The two auctions are revenue-equivalent and payoff-equivalent if the surplus-extracting continuation fee is imposed.*

Now that we have established payoff and revenue equivalence across auctions with symmetric and increasing equilibria plus identical values of $\widehat{t}$, we conclude by characterizing the auctioneer's optimal selection of $\widehat{t}$ to maximize revenue. The following proposition establishes that, if the strong auctioneer can charge the surplus-extracting continuation fee, the allocation of the prize in a revenue-maximizing charity auction is the same as in the revenue-maximizing non-charity auction. This follows because, framed as mechanism-design problems, the charity and non-charity auctions have the same constraints and the objective function is the same up to a factor of proportionality, by equation (20). The constraints are incentive compatibility, which amounts to the probability of winning the prize being nondecreasing in one's type, and individual rationality, which ensures that each bidder's expected payoff is at least zero, which is the payoff from not participating. As in the non-charity case, the optimal $\widehat{t}$ ensures that the winner has a positive virtual valuation.

Proposition 6. *The strong auctioneer maximizes revenue by: 1) allocating the prize exactly as in the non-charity optimal auction, and 2) charging the surplus-extracting continuation fee.*

This result implies that there is not just one optimal auction in this context but many. Goeree *et al.* (2005) point out that the auctioneer's expected revenue is maximized when all bidders pay a suitable continuation fee, and then take part in an auction in which there is an appropriately-chosen reserve price, and all those who choose to bid pay the lowest bid or the reserve. However, contrary to what they suggest, this is only one of many optimal auctions. For example, a first-price, second-price or all-pay auction supplemented by the appropriate bidding fee (to set $\widehat{t}$ optimally) and continuation fee (to ensure that the lowest types receive no surplus) are all optimal auctions when the auctioneer is strong.

4.4 Revenue with a Weak Auctioneer

In an auction with a weak auctioneer, bidders with types above a threshold $\widehat{t}$ submit bids above r and also pay φ. The auctioneer is unable to charge an additional continuation fee that is paid by all bidders, including those who do not submit bids. We begin our approach to the weak auctioneer's revenue-maximization problem by considering whether the auctioneer has a preference between using r or φ to achieve a given threshold value of $\widehat{t}$. The values of r and φ affect bidder utility, and in first-price and all-pay auctions they affect the bids themselves. We find that bidding fees raise at least as much revenue as reserve prices across all auction formats and for any $\widehat{t} \in (\underline{t}, \overline{t})$. This limits the impact of our assumption that in first- and second-price auctions r must be low enough to ensure a separating equilibrium.

Proposition 7: *In first- and second-price auctions with a weak auctioneer and strictly increasing bid functions, the revenue-maximizing way to implicitly select a threshold bidder type $\widehat{t} \in (\underline{t}, \overline{t})$ is to set $\varphi > 0$ and $r = 0$. In an all-pay auction φ and r are interchangeable; conditional on any $\widehat{t}$ revenue is constant across all appropriate combinations of r and φ.*

In the first-price auction, a small decrease in r and an increase in φ to preserve $\widehat{t}$ has no effect on the expected payment of the threshold bidder, $x_1(\widehat{t})$. However, the equilibrium bidding function $B_1(t|r,\varphi)$ prescribes that bidders with types above $\widehat{t}$ decrease their bids by less than the change in r while paying the full increase in φ. This increases the expected payments of bidders with $t > \widehat{t}$, so expected revenue ultimately rises. The second-price auction is different in that r and φ do not affect equilibrium bidding by active bidders. Instead, the important feature of this auction is that a reduction in r can be matched with a relatively large increase in φ because of a losing bidder's role in setting the winner's price. All bidders with types above $\widehat{t}$ pay this (large) increase in φ, so the revenue-maximizing auctioneer will choose to reduce r to zero and set a positive φ. In an all-pay charity auction, we obtain the familiar result from non-charity auctions. This is as expected, since bidding is affected only through the shift factor $\frac{1}{1-\theta}$. Given that we know that bidding fees are preferred to reserve prices (weakly so for all-pay auctions), we continue with our revenue analysis under the assumption that φ may be greater than zero and $r = 0$ in all auctions.

Next, we demonstrate that the weak auctioneer's expected revenue is bounded and can be expressed in a way that facilitates revenue comparisons. The boundedness result is noteworthy because a strong auctioneer can raise unlimited revenue when $\Omega < 0$.

Proposition 8: *Revenue from each auction is bounded and can be expressed as*

$$R_g = \left(\frac{1}{1-\theta}\right) \int_{\widehat{t}}^{\overline{t}} \left\{ t - \frac{[1-F(t)]}{f(t)} \right\} dF_1^n(t) - \frac{\lambda \delta_g}{1-\theta},$$

with

$$\begin{aligned}
\delta_1 &= \int_{\widehat{t}}^{\overline{t}} B_1(t|r,\varphi)[ndF_1^{n-1}(t) - (n-1)dF_1^n(t)] = \int_{\widehat{t}}^{\overline{t}} B_1(t|r,\varphi) dF_2^n(t) \\
\delta_2 &= \int_{\widehat{t}}^{\overline{t}} B_2(t)[ndF_2^{n-1}(t) - (n-1)dF_2^n(t)] \\
\delta_A &= 0.
\end{aligned}$$

This result on bounded auction revenue follows from the properties of the δ_g terms, which are bounded themselves. Our revenue comparisons depend on the characteristics of the δ_gs, and are provided in the following Corollary.

Corollary 2: *Fix* $\widehat{t} \in [\underline{t}, \overline{t})$. $R_A > R_1$ *for all* n. $R_2 > R_A$ *for* $n = 2$, *but* $R_A > R_2$ *for* n *sufficiently large. As* $n \to \infty$, R_2 *and* R_1 *converge.*

Two results in this Corollary are essentially extensions of our results on absolute auctions: 1) the ranking of R_A and R_1, and 2) the revenue ranking as $n \to \infty$. However, our result in Corollary 2 on second-price auctions with $n = 2$ is notable in its difference from the absolute auction, in which the revenue ranking of second-price and all-pay auctions is ambiguous. Let $\varphi_g(\widehat{t})$ represent the maximum fee that the weak auctioneer can charge while insuring participation of a type-$\widehat{t}$ bidder in a format-g auction. In all-pay and first-price auctions, $\varphi_g(\underline{t}) = 0$ and the weak auctioneer's expected revenue is identical to that of the absolute auction. However, as noted in Section 3.1.3, in a second-price auction with $n = 2$ the bidder with type $\underline{t}$ is willing to participate in an auction with positive φ because of her (certain) role in setting the winner's payment. The maximum fee a bidder with type $\underline{t}$ will pay in a two-bidder second-price auction is $\varphi_2(\underline{t}) = \frac{\lambda B_2(\underline{t})}{1-\theta}$, which equates the utility from participating in the auction and opting out. Across auctions, if the auctioneer seeks to improve revenue by increasing $\widehat{t}$, he can do so by raising the fee above $\varphi_g(\underline{t})$.

Finally, we characterize how the weak auctioneer makes an optimal choice of $\widehat{t}$ (through φ) in each auction.

Proposition 9: *The revenue-maximizing threshold value* $\widehat{t}$ *for an all-pay auction is set just as in a non-charity auction, while the optimal first-price charity auction includes fewer bidders in expectation. The optimal threshold for a second-price auction is lower than the all-pay optimum* *for $n = 2$, but for a sufficiently large n the second-price auction includes fewer active bidders than the all-pay auction.*

Since all-pay revenue differs from the non-charity case only in the scale factor $(1-\theta)^{-1}$, it is not surprising that the optimal $\widehat{t}$ in an all-pay auction is exactly the non-charity solution. That is, the optimal $\widehat{t}$ is identified by simply excluding any bidders with a negative virtual valuation. In a first-price auction, there is an additional benefit to increasing $\widehat{t}$ above the all-pay optimum because δ_1 is decreasing in the threshold value. The result on second-price auctions for $n = 2$ follows from the important role of the low bidder in setting the winner's price. This role provides bidders with additional utility from participation, and this reduces the "cost" to the auctioneer of extending $\widehat{t}$ downward though a reduction in φ. As n grows, the probability that a bidder with a low t determines the winner's payment falls, so the auctioneer's incentive to expand downward the set of active bidders diminishes.

5 Conclusions

This paper compares equilibrium bidding and revenue in three charity auction formats: first-price, second-price, and all-pay. We find that participants in all types of auctions bid higher when they benefit from revenue collected by the charity. If bidders receive no benefit from the payments by other bidders, they simply inflate proportionally the bids that they would have made in a standard (non-charity) auction. Bidding incentives are more complicated when auction revenue is like a public good that all bidders enjoy regardless of who contributes. An increase in the benefit from others' payments can depress bids in the single-price auctions, and revenue typically varies across absolute auction formats. All-pay auctions have the greatest revenue among absolute auction formats when the number of bidders is sufficiently large, and second-price auction revenue exceeds first-price revenue.

If the auctioneer can set fees and a reserve price, some bidders may want to refrain from bidding, and the consequences of non-participation will affect revenue. If the auctioneer is strong, meaning he can cancel the auction unless he receives the requisite payments from all potential bidders, the same expected revenue can be raised in each of the three bidding formats considered here, and indeed we show that a general revenue equivalence result holds in this case. Thus, with appropriate fixed fees, the standard auction formats considered in this paper are revenue-equivalent to the optimal auction proposed by Goeree *et al.* (2005), in which all bidders pay the reserve price, or if all bidders bid above the reserve, the lowest of the bids.

Because such a strong commitment ability to cancel the auction seems extreme, we also consider charity auctions in which bidders can opt out yet receive utility from the payments of others, much as nonpayers can derive benefit from a public good because it is nonexcludable. In this case, revenue equivalence no longer holds. Our consideration of a weak auctioneer is distinct from the model of a constrained auctioneer studied by Goeree *et al.* In their analysis, an auctioneer may be unable to commit to strategies that could allow the object to go unsold (*i.e.*, fixed fees or reserve prices), and Goeree *et al.* find that the optimal auction in this case is the one in which all bidders pay the lowest bid. However, this auction does not permit the bidders to opt out without eliminating all auction revenue. A missing bid is interpreted as a zero bid and hence leads to zero revenue collected and zero utility, just as in the case when the auctioneer can cancel the auction. If the constrained auctioneer were to collect any revenue when some bidders opt out, bidders with sufficiently low valuations of the prize would prefer to refrain from bidding rather than make their equilibrium bid, which is near zero.

Issues left for future research include characterizing the optimal auction when potential bidders are free to opt out without dissolution of the auction, empirical and experimental research on charity auctions,[20] and determining when a charitable organization would prefer to hold an auction instead of using a different fund-raising method. It may be the case that auctions are most useful when potential contributors are particularly unwilling to open their wallets for a charity.

[20]See Davis *et al.* (2003) and Isaac and Schnier (2003) for expermental evidence on the practical implementation of charity auctions.

A Appendix 1: Proofs and analysis for bidding and utility

We begin with two lemmas that establish attributes of the bidding strategies described in the body of this paper. First, we show that bidders prefer their own proposed equilibrium strategies to those of other bidders. Second, we show that active bidders prefer to participate in the auction rather than opt out entirely. In the strong auctioneer case, this result also extends to inactive bidders who pay c.

Lemma A1: A bidder with type $t \in [\widehat{t}, \overline{t}]$, where $\widehat{t}$ may equal $\underline{t}$, prefers signalling her own type rather than follow the proposed equilibrium bidding strategy of any other bidder with distinct $s \in [\underline{t}, \overline{t}]$. A bidder with type $t \in [\underline{t}, \widehat{t})$, if such types exist, prefers to announce "no bid" and receive $U_g^{out}(\widehat{t})$ rather than imitate any active bidder of type $s \in [\widehat{t}, \overline{t}]$.

Proof of Lemma A1: When a bidder of type $t \in [\widehat{t}, \overline{t}]$ considers imitating the strategy of a type $s \in [\underline{t}, \overline{t}]$ bidder, there are two cases to consider. First, if $s \geq \widehat{t}$, the return to the type-t bidder from announcing s is

$$\begin{aligned} \pi(s|t) &= (t-s)F(s)^{n-1} + \pi(s|s) \\ &= (t-s)F(s)^{n-1} + U(s), \end{aligned}$$

where $U(s)$ denotes $\pi(s|s)$. Because the first-order condition for a bidder of type t is satisfied we can apply the envelope theorem to obtain $U'(t) = F(t)^{n-1}$. This establishes that the function U is increasing and strictly convex because its derivative, $F(t)^{n-1}$, is positive and strictly increasing in t. The strict convexity of the utility function implies $(t-s)U'(s) + U(s) < U(t)$ for any $t \in [\underline{t}, \overline{t}]$ and $s \in (\widehat{t}, \overline{t})$, with $s \neq t$. Then, since $U'(t) = F(t)^{n-1}$, we may write $(t-s)F(s)^{n-1} + U(s) < U(t)$. This implies $\pi(s|t) < \pi(t|t)$ for any distinct s and t with $s \in (\widehat{t}, \overline{t})$. Values of s at the end points of $[\widehat{t}, \overline{t}]$ may be ruled out too. For $s = \widehat{t}$, $U(\widehat{t}) < U(t)$ because U is increasing. Similarly, for $s = \overline{t}$, the inequality is equivalent to $t - \overline{t} + U(\overline{t}) < U(t)$ which is true because $U'(t) < 1$ for $t < \overline{t}$. This establishes that the signalling strategy which satisfies condition (2) is optimal.

The second case for a bidder with $t \geq \widehat{t}$ is when $s < \widehat{t}$. By the definition of $\widehat{t}$ for each auction format g, $U_g^{in}(\widehat{t}) \geq U_g^{out}(\widehat{t})$, so $U_g(t) \geq U_g^{out}(\widehat{t})$ and the type-t bidder is content to follow the proposed equilibrium strategy.

Next consider whether a bidder whose type is in $[\underline{t}, \widehat{t})$ can benefit from imitating a bidder with type $s \in [\widehat{t}, \overline{t}]$. First, we note that following the proposed equilibrium strategy is preferred to imitating a bidder of type $s = \widehat{t}$, since the additional surplus from doing this (on top of the equilibrium payoff of $U_g^{out}(\widehat{t})$) is $F(\widehat{t})^{n-1}(t - \widehat{t}) < 0$. Further, the bidder's return from increasing s above $\widehat{t}$ reduces utility, since

$$\frac{d\pi(s|t)}{ds} = (t - s)\frac{dF(s)^{n-1}}{ds} + \frac{d\pi(s|s)}{ds} < 0.$$

This inequality holds because $t < s$ and $\frac{d\pi(s|s)}{ds} = 0$ in the proposed equilibrium for $s \in (\widehat{t}, \overline{t})$. Since we have assumed that the bid functions are continuous over the support of t, it is also true that the type-t bidder prefers to announce her true type rather than $\overline{t}$. **Q.E.D.**

Lemma A2: *In an absolute auction all bidders prefer to participate in the auction rather than announce "no bid." In an auction with a weak auctioneer, bidders with types at or above $\widehat{t}$ prefer their posited equilibrium bids to announcing "no bid." Finally, in an auction with a strong auctioneer and sufficiently low values of c, no bidder will unilaterally deviate from the situation in which all bidders pay c and those with types in $[\widehat{t}, \overline{t}]$ bid according to $B_g(\cdot|r, \varphi)$.*

Proof of Lemma A2: In the absolute auction, when bidder i unilaterally opts out of the auction while all other bidders use B_g, bidder i receives utility equal to $U(\underline{t}) \geq 0$. We established in the proof of Lemma A1 that $U(t)$ is increasing, so a bidder with $t \in [\underline{t}, \overline{t}]$ weakly prefers the equilibrium payoff of $U(t)$ to the utility from opting out. Similarly, in the weak auctioneer case if any bidder opts out she receives $U_g^{out}(\widehat{t}) \leq U_g^{in}(\widehat{t})$, which is no greater than $U(t)$ for $t \in [\widehat{t}, \overline{t}]$ since U is increasing. The same argument establishes that in the strong auctioneer case a bidder with type $t \in [\widehat{t}, \overline{t}]$ prefers paying c and following $B_g(\cdot|r, \varphi)$ rather than paying c and announcing "no bid." No bidder will defect from paying c if $U_g^{out}(\widehat{t})$, the minimum utility from paying the continuation fee, is nonnegative, since any defection leads to dissolution of the auction and zero utility for all bidders. Since $U_g^{out}(\widehat{t}) > 0$ for each auction format when $c = 0$, positive values of c exist such that $U_g^{out}(\widehat{t}) \geq 0$. **Q.E.D.**

These lemmas apply to each of the auction formats (first-price, second-price, and all-pay), and the cases of an absolute auction, a strong auctioneer, and a weak auctioneer. Therefore in the remaining sections of this appendix we focus on aspects of the equilibria that are particular to a selling format or set of auction rules. In particular, for each auction we demonstrate that the equilibrium bidding function is increasing in t, which is a necessary condition for the existence of an increasing equilibrium.

A.1 First-price auctions

In the absolute auction, we verify that the bidding function B_1 is increasing by examining the derivative of B_1,

$$B_1'(t) = \left(\frac{\alpha}{1-\Delta}\right)\frac{f(t)}{F(t)}\int_{\underline{t}}^{t}\left(\frac{F(x)}{F(t)}\right)^{\alpha}dx.$$

$B_1'(t)$ exists for all $t \in (\underline{t}, \bar{t})$ and is strictly positive. For a strong or weak auctioneer, we verify that $B_1(t|r,\varphi)$ is strictly increasing for all $t \in (\widehat{t}, \bar{t})$ by noting that

$$B_1'(t|r,\varphi) = B_1'(t|0,0) + \left(\frac{\widehat{t}}{1-\Delta} - r\right)\alpha f(t)\frac{F(\widehat{t})^{\alpha}}{F(t)^{\alpha+1}}$$

is positive for $t \in (\widehat{t}, \bar{t})$ given that $\widehat{t} \geq (1-\Delta)r$.

A.2 Second-price auctions

In a separating equilibrium among active bidders, the bidding function of a second-price auction is unaffected by the level of a reserve or fees. Thus, we demonstrate that the common bidding function B_2 is increasing in t by examining its derivative,

$$B_2'(t) = \left(\frac{\beta}{1-\Delta}\right)\left(\frac{f(t)}{1-F(t)}\right)\int_{t}^{\bar{t}}\left(\frac{1-F(x)}{1-F(t)}\right)^{\beta}dx.$$

B_2' exists for all $t \in (\underline{t}, \bar{t})$ and is strictly positive.

Next, we prove Lemma 1 and Proposition 1, which describe, respectively, properties of the bidding function B_2 and an equilibrium exit strategy for bidders in a button auction.

Proof of Lemma 1. Part (a) is immediate since the bid function B_2 does not depend on the number of bidders. Part (b) is verified by substituting $\frac{F(s)-F(x)}{1-F(x)}$ for $F(s)$ in B_2 and noting that the substitution leaves B_2 unaltered. **Q.E.D.**

Proof of Proposition 1. We begin with a simplified static game that operates in the same way as the button auction except that, before the price starts to rise above zero, all bidders choose simultaneously and once-and-for-all the price at which their button is to be released. The rules of the button auction mean that this static game has exactly the same payoff structure as the second-price sealed-bid auction, and hence it is a Bayesian Nash equilibrium for each bidder of type t to release her button at the price $B_2(t)$.

Next consider case (a): a dynamic game in which bidders can decide whether to exit as the price indicator rises, and bidders can observe when rivals exit. Choose any bidder i of type t and suppose that all bidders other than i follow the posited equilibrium strategy, using B_2 to determine their exit prices. We associate with each of i's information sets beliefs that are consistent with rivals' strategies and Bayes' Rule along the equilibrium path, and show that, given the beliefs, i's equilibrium choice at each information set is optimal. Each information set corresponds to a price p that has been reached and the history of prices at which exit occurred. Because values are private and independent, the only payoff-relevant aspect of the history is the number of remaining rivals m. We must specify beliefs about the types of these rivals. For prices $p < B_2(\underline{t})$, posterior beliefs are the same as prior beliefs, that the types of the remaining rival bidders are independently drawn from distribution F. For any price $p \in [B_2(\underline{t}), B_2(\bar{t}))$, let $x = B_2^{-1}(p)$. Then i's posterior belief is that the remaining rivals' types s are drawn independently from the distribution $F(s|s \geq x)$. Prices $p \geq B_2(\bar{t})$ are reached with zero probability along the equilibrium path, and arbitrary beliefs with support in $[\underline{t}, \bar{t}]$ will meet the requirements of equilibrium.

Now we show that, given these beliefs, i will find it optimal to exit at $p \in [B_2(\underline{t}), B_2(\bar{t}))$ if and only if $p \geq B_2(t)$ (or equivalently, if $t \leq x = B_2^{-1}(p)$) We first show that it is optimal for i to exit at p if $t = x$. The reason is that at p, i's expected payoff as a function of its exit price b is exactly the same as i's payoff in a second-price auction with bid b, in which there are m other bidders and the rivals each have types drawn independently from the distribution $F(s|s \geq x)$. Lemma 1 implies that the bid function B_2 gives a Bayesian Nash equilibrium in this auction with any number of bidders and with this truncated distribution. Thus type x finds it optimal to exit immediately rather than wait. Types $t > x$ do better by waiting until $B_2(t)$ and so their equilibrium choice of not exiting at p is optimal. Types $t < x$ would achieve the same payoff as x from immediate exit, while waiting (which leads to a positive probability of winning) raises type t's payoff by less than it would raise x's. Since exit is optimal for x, it is optimal for $t < x$. Next, exit at any price $p < B_2(\underline{t})$, yields the same payoff as exit at $p = B_2(\underline{t})$, so it is optimal to continue rather than exit at such prices. Finally, if $p > B_2(\bar{t})$, any type of bidder prefers to lose rather than win, and the other remaining bidders' strategies posit immediate exit, so immediate exit is optimal for i. This completes the proof of case (a)

Now consider case (b) in which bidders do not observe others' exit decisions. The beliefs at an information set also provide a probability distribution over the *number* of remaining rivals as well as their types. For prices $p < B_2(\underline{t})$ the number is believed to be $n-1$ with certainty. For $p > B_2(\bar{t})$ arbitrary beliefs suffice. For any price $p \in [B_2(\underline{t}), B_2(\bar{t})]$, let $x = B_2^{-1}(p)$ as before. Then the belief about the number of rivals is given by a conditional binomial distribution: the number of successes in $n-1$ Bernoulli trials with probability of success $1-F(x)$, all conditional on there being at least one success, (or else the auction would have terminated).

As above, at any information set we can determine bidder i's expected payoff from immediate exit and compare it with the expected payoff from delaying until a later exit price. In case (a) we showed that, given the rivals' equilibrium strategies, the posited strategy is optimal regardless of how many other bidders remain. At each information set the payoffs now are just a convex combination of those in (a) where the weights are constant. If a set of functions are maximized at the same point, then a convex combination of these functions is also maximized at this point. Hence, in case (b) too, the strategies comprise a Perfect Bayesian Equilibrium. **Q.E.D.**

Finally, we show in an auction with a weak or strong auctioneer, equation (11) uniquely determines the binding $\widehat{t}$ conditional on the bidding function B_2 and $B_2(\widehat{t}) \geq r$. We write the indifference condition as $g(\widehat{t}) = 0$, with

$$g(\widehat{t}) = F(\widehat{t})^{n-1}[\widehat{t} - (1-\theta)r] - (1-\theta)\varphi + \lambda(n-1)F(\widehat{t})^{n-2}[1-F(\widehat{t})][B_2(\widehat{t}) - r].$$

The function g has the following properties. When $n > 2$, $g(\underline{t}) = -(1-\theta)\varphi$, and when $n = 2$, $g(\underline{t}) = \lambda[B_2(\underline{t}) - r] - (1-\theta)\varphi$. Also, $g(\overline{t}) = \overline{t} - (1-\theta)(r+\varphi)$. Next, we show that g is increasing in $\widehat{t}$ by examining its derivative:

$$g'(\widehat{t}) = [B_2(\widehat{t}) - r]\left[(1-\theta)\frac{dF_1^{n-1}(\widehat{t})}{d\widehat{t}} + \lambda\frac{dF_2^{n-1}(\widehat{t})}{d\widehat{t}}\right] + F(\widehat{t})^{n-1} + \lambda r\frac{dF_1^{n-1}(\widehat{t})}{d\widehat{t}} > 0.$$

In this expression for g' we use the notational convention $\frac{dF_2^1}{dt} = 0$ to incorporate the $n = 2$ case, and we use equation (9) to eliminate B_2' from g'. The sign of g' follows from the restriction that $B_2(\widehat{t}) \geq r$ under the rules of the auction. The auctioneer must set r and φ such that $g(\overline{t}) > 0$, or else no bidder will ever participate in the auction. Thus when $n > 2$, $g(\widehat{t}) = 0$ at a unique $\widehat{t}$ since $g(\underline{t}) < 0$, $g(\overline{t}) > 0$, and $g'(t) > 0$. The $n = 2$ case is different only in that $g(\underline{t})$ may be greater than zero. If this is the case, all bidders participate in the auction; otherwise, g again determines a unique value of $\widehat{t}$.

A.3 All-pay auctions

In the absolute auction case, the bidding function B_A is increasing in t because

$$B_A'(t) = \frac{t}{1-\theta}\frac{dF(t)^{n-1}}{dt}$$

exists for all $t \in (\underline{t}, \overline{t})$ and is strictly positive. In all-pay auctions with a weak or strong auctioneer, the derivative of $B_A(t|r,\varphi)$ is the same as when $r = \varphi = 0$.

B Appendix 2: Proofs and analysis for the revenue results

B.1 Absolute auctions

Proof of Proposition 2. For each auction format we begin with equation (20) with $\widehat{t} = \underline{t}$. Differences in bidding revenue are captured entirely by the term $U_g(\underline{t})$. When $n > 2$, these boundary values of utility are

$$\begin{aligned} U_1(\underline{t}) &= \lambda \int_{\underline{t}}^{\overline{t}} B_1(t) dF_1^{n-1}(t) \\ U_2(\underline{t}) &= \lambda \int_{\underline{t}}^{\overline{t}} B_2(t) dF_2^{n-1}(t) \\ U_A(\underline{t}) &= \lambda(n-1) \int_{\underline{t}}^{\overline{t}} B_A(t) f(t) dt. \end{aligned}$$

Note that these bid functions are those that apply to auctions with n bidders. However, when the bidder with type $\underline{t}$ considers utility (and revenue) from the auction, she considers the actions of the $(n-1)$ other bidders. If $n = 2$, the expressions for $U_1(\underline{t})$ and $U_A(\underline{t})$ are unchanged, but $U_2(\underline{t})$ becomes $\lambda B_2(\underline{t})$, since the bidder of type $\underline{t}$ is sure to lose the auction but also determine the payment made by the winner. It immediately follows from the expressions above that the auction formats are revenue-equivalent if $\lambda = 0$.

The expression for revenue in an all-pay absolute auction may be simplified due to the form of $U_A(\underline{t})$. Since $U_A(\underline{t}) = \lambda(n-1)X_A$, we use the identity $\Omega R_A = [1 - \theta - \lambda(n-1)]nX_A$ to write

$$R_A = \left(\frac{1}{1-\theta}\right) \int_{\underline{t}}^{\overline{t}} \left\{ t - \frac{[1-F(t)]}{f(t)} \right\} dF_1^n dt.$$

Now consider whether similar simplifications may be made for first- and second-price absolute auctions. In a first-price auction the *ex ante* expected payment of a bidder is $X_1 = \frac{1}{n} \int_{\underline{t}}^{\overline{t}} B_1(t) dF_1^n(t)$. We define $\widetilde{X}_1$ such that

$$\begin{aligned} \lambda(n-1)\widetilde{X}_1 &= U_1(\underline{t}) \\ &= \lambda(n-1) \int_{\underline{t}}^{\overline{t}} B_1(t) F(t)^{n-2} f(t) dt \\ &= \lambda(n-1)X_1 + \lambda(n-1) \int_{\underline{t}}^{\overline{t}} B_1(t) \left[F(t)^{n-2} - F(t)^{n-1} \right] f(t) dt. \end{aligned}$$

Notice that $n\lambda(n-1)\widetilde{X}_1 = n\lambda(n-1)X_1 + \lambda\int_{\underline{t}}^{\overline{t}} B_1(t)dF_2^n(t)$. We use the relationship between $\widetilde{X}_1$ and $U_1(\underline{t})$ to write

$$\Omega R_1 = \int_{\underline{t}}^{\overline{t}} \left\{ t - \frac{[1-F(t)]}{f(t)} \right\} dF_1^n dt - n\lambda(n-1)\widetilde{X}_1.$$

Both ΩR_1 and $n\lambda(n-1)\widetilde{X}_1$ contain the term $n\lambda(n-1)X_1$, and once this term is eliminated from each side of the equation above we have

$$R_1 = \left(\frac{1}{1-\theta}\right) \int_{\underline{t}}^{\overline{t}} \left\{ t - \frac{[1-F(t)]}{f(t)} \right\} dF_1^n dt - \frac{\lambda}{1-\theta} \int_{\underline{t}}^{\overline{t}} B_1(t)dF_2^n(t).$$

Therefore, $R_1 < R_A$ when $\lambda > 0$, since $\int_{\underline{t}}^{\overline{t}} B_1(t)dF_2^n(t)$, the expected value of the second-highest of n bids, is positive in a first-price auction.

The characteristics of $U_2(\underline{t})$ and $\lambda(n-1)X_2$ prevent an unambiguous ranking of these terms. We show in Example 1 in the text that the ranking of R_2 and R_A depends on parameter values. **Q.E.D.**

The following lemma is necessary for the proof of Proposition 3 but holds no economic content, so it is presented in this appendix only. The proof of Proposition 3 follows.

Lemma A3. Suppose that $z : [a,b] \to \Re$ is continuous on $[a,b]$, differentiable on (a,b), and $z = 0$ at a and b. If z is positive wherever its derivative vanishes, then z is positive on (a,b). Alternatively, if z is negative wherever $z' = 0$, then z is negative on (a,b).

Proof of Lemma A3. We prove the first part of the lemma (which concludes z is positive on (a,b)) by proving the contrapositive. Suppose $z(y) \leq 0$ for some s in (a,b). Then either s is a global minimum or there is a global minimum m such that $z(m) < z(s) \leq 0$. In either case, we have found a global minimum in (a,b). As this is an interior local minimum, z' must vanish at a point where z is not positive. A similar argument proves the second part of the lemma. **Q.E.D.**

Proof of Proposition 3: In Case 1 of this proof we suppose that $\Omega \neq 0$, and the revenue result is established by examination of $x_2(\overline{t}) - x_1(\overline{t})$. In Case 2 we assume $\Omega = 0$ and prove $R_2 > R_1$ directly.

Case 1: Our first step is to rewrite $x_2(\overline{t})$ as

$$\begin{aligned} x_2(\overline{t}) &= \frac{1}{1-\Delta} \left\{ \int_{\underline{t}}^{\overline{t}} t dF_1^{n-1}(t) + \int_{\underline{t}}^{\overline{t}} \int_t^{\overline{t}} \left[\frac{1-F(s)}{1-F(t)}\right]^\beta ds dF_1^{n-1}(t) \right\} \\ &= \frac{1}{1-\Delta} \left\{ \overline{t} - \int_{\underline{t}}^{\overline{t}} F(t)^{n-1} dt + \int_{\underline{t}}^{\overline{t}} \int_{\underline{t}}^{s} \frac{dF(t)^{n-1}}{[1-F(t)]^\beta} [1-F(s)]^\beta ds \right\}. \end{aligned}$$

Now the difference in expected prices depends on the value of an expression integrated from $\underline{t}$ to $\overline{t}$

$$x_2(\overline{t}) - x_1(\overline{t}) = \frac{1}{1-\Delta}\int_{\underline{t}}^{\overline{t}} \left\{ \left(\int_{\underline{t}}^{s} \frac{dF(t)^{n-1}}{[1-F(t)]^{\beta}} \right) [1-F(s)]^{\beta} + F(s)^{\alpha} - F(s)^{n-1} \right\} ds$$

Let $g(s)$ represent the terms in braces, so that

$$x_2(\overline{t}) - x_1(\overline{t}) = \frac{1}{1-\Delta}\int_{\underline{t}}^{\overline{t}} g(s)ds.$$

We verify the sign of $x_2(\overline{t}) - x_1(\overline{t})$ by showing that the integrand g is either positive $\forall\ s$ or negative $\forall\ s$, depending on parameter values and the number of bidders.

We now demonstrate that Lemma A3 applies to g. The continuity and differentiability of g follow from the assumption that F has these properties; substitution shows that $g(\overline{t}) = g(\underline{t}) = 0$. Differentiation yields

$$g'(s) = \alpha F(s)^{\alpha-1} f(s) - \left(\int_{\underline{t}}^{s} \frac{dF(t)^{n-1}}{[1-F(t)]^{\beta}} \right) \beta[1-F(s)]^{\beta-1} f(s).$$

And since the density f is positive

$$g'(s) = 0 \Leftrightarrow \left(\int_{\underline{t}}^{s} \frac{dF(t)^{n-1}}{[1-F(t)]^{\beta}} \right) = \frac{\alpha F(s)^{\alpha-1}}{\beta[1-F(s)]^{\beta-1}}.$$

Consider an $s \in (\underline{t}, \overline{t})$ at which $g'(s) = 0$. We now show that at any such s, $g(s) > 0$ if $\Omega > 0$ and $g(s) < 0$ if $\Omega < 0$. When $g'(s) = 0$ it must be true that

$$g(s) = \frac{\alpha}{\beta} F(s)^{\alpha-1}[1-F(s)] + F(s)^{\alpha} - F(s)^{n-1}.$$

The sign of $g(s)$ is unaffected by division by $F(s)^{n-1}$, which we use to define $h = g/F^{n-1}$ and write

$$h(s) = qF(s)^{q-1} + [1-q]F(s)^{q} - 1,$$

where $q = \lambda(n-1)/(1-\theta)$. The new term q takes values in $(0, \infty)$, and the relevant cases for this term are when $q < 1$ ($x_2(\overline{t}) - x_1(\overline{t})$ and $R_2 - R_1$ have the same sign) and $q > 1$ ($x_2(\overline{t}) - x_1(\overline{t})$ and $R_2 - R_1$ have the opposite sign). The case of $q = 1$, which occurs when $\Omega = 0$, is addressed below.

Our approach is to show that the sign of h is positive (negative) for all $s \in (\underline{t}, \overline{t})$ when $q < 1$ ($q > 1$). If h has the appropriate sign for all s, then g will have the same sign as h whenever $g' = 0$. First note that when $q < 1$, it is true that $\lim_{s \to \underline{t}} h(s) = \infty$, $\lim_{s \to \overline{t}} h(s) = 0$, and $h'(s) < 0$ $\forall\ s \in (\underline{t}, \overline{t})$. Together, these conditions imply that $h(s) > 0\ \forall\ s \in (\underline{t}, \overline{t})$ when $q < 1$, so we may also conclude that $g > 0$ when $g' = 0$. By Lemma A3, this implies $g > 0$ for all $s \in (\underline{t}, \overline{t})$, so $x_2(\overline{t}) > x_1(\overline{t})$ and $R_2 > R_1$ when $q < 1$. Now consider $q > 1$, for which $\lim_{s \to \underline{t}} h(s) = -1$, $\lim_{s \to \overline{t}} h(s) = 0$, and $h'(s) > 0\ \forall\ s \in (\underline{t}, \overline{t})$. These conditions imply $h(s) < 0\ \forall\ s \in (\underline{t}, \overline{t})$ when $q > 1$, so $g < 0$ when $g' = 0$. By Lemma A3, this implies $g < 0$ for all $s \in (\underline{t}, \overline{t})$, so $x_2(\overline{t}) < x_1(\overline{t})$. When $q > 1$ have $x_2(\overline{t}) < x_1(\overline{t}) \Rightarrow R_2 > R_1$.

Case 2: The maintained assumption of $\Omega = 0$ implies that the parameters α and β, which appear in B_1 and B_2, are both equal to n. This simplifies expected revenue terms and facilitates their direct comparison. These terms are:

$$\begin{aligned}
R_1 &= \frac{1}{1-\Delta}\int_{\underline{t}}^{\overline{t}}\left\{t - \int_{\underline{t}}^{t}\left[\frac{F(s)}{F(t)}\right]^n ds\right\} dF_1^n(t) \\
&= \frac{1}{1-\Delta}\left\{\int_{\underline{t}}^{\overline{t}} t dF(t)^n - \int_{\underline{t}}^{\overline{t}}\int_{s}^{\overline{t}} \frac{dF(t)^n}{F(t)^n} F(s)^n ds\right\} \\
&= \frac{1}{1-\Delta}\left\{\overline{t} - \int_{\underline{t}}^{\overline{t}} F(t)^n dt + n\int_{\underline{t}}^{\overline{t}} \left[\log F(t)\right] F(t)^n dt\right\} \\
R_2 &= \frac{1}{1-\Delta}\int_{\underline{t}}^{\overline{t}}\left\{t + \int_{t}^{\overline{t}}\left[\frac{1-F(s)}{1-F(t)}\right]^n ds\right\} dF_2^n(t) \\
&= \frac{1}{1-\Delta}\left\{\int_{\underline{t}}^{\overline{t}} t dF_2^n(t) + \int_{\underline{t}}^{\overline{t}}\int_{\underline{t}}^{s} \frac{dF_2^n(t)}{[1-F(t)]^n}[1-F(s)]^n ds\right\} \\
&= \frac{1}{1-\Delta}\left\{\overline{t} - \int_{\underline{t}}^{\overline{t}} F_2^n(t) dt + \int_{\underline{t}}^{\overline{t}}\int_{\underline{t}}^{s} \frac{dF_2^n(t)}{[1-F(t)]^n}[1-F(s)]^n ds\right\},
\end{aligned}$$

Next, define the function $g(s)$ so that

$$R_2 - R_1 = \frac{1}{1-\Delta}\int_{\underline{t}}^{\overline{t}} g(s) ds.$$

From the expected revenue expressions above we see that

$$g(s) = \int_{\underline{t}}^{s} \frac{dF_2^n(t)}{[1-F(t)]^n}[1-F(s)]^n - nF(s)^{n-1}[1-F(s)] - n\left[\log F(s)\right] F(s)^n.$$

Differentiation of g yields

$$g'(s) = -\int_{\underline{t}}^{s} \frac{dF_2^n(t)}{[1-F(t)]^n} n[1-F(s)]^{n-1} f(s) - n^2\left[\log F(s)\right] F(s)^{n-1} f(s).$$

This derivative is equal to zero when

$$\int_{\underline{t}}^{s} \frac{dF_2^n(t)}{[1-F(t)]^n} = -\frac{n\left[\log F(s)\right] F(s)^{n-1}}{[1-F(s)]^{n-1}}.$$

Consider an $s \in (\underline{t}, \overline{t})$ at which $g'(s) = 0$. We now show that at any such s, $g(s) > 0$. When $g'(s) = 0$ it must be true that

$$g(s) = -n\left[\log F(s)\right] F(s)^{n-1} - nF(s)^{n-1}[1-F(s)].$$

The sign of this expression is unaffected when we divide by $nF(s)^{n-1}$, so we do this to obtain the simpler expression

$$h(s) = -\log[F(s)] - [1-F(s)].$$

As in Case 1, we show that the sign of h is constant for all s in order to establish the sign of g whenever $g' = 0$. The function h has the following properties: $\lim_{s \to \underline{t}} h(s) = \infty$, $\lim_{s \to \overline{t}} h(s) = 0$, and $h'(s) < 0 \; \forall \; s \in (\underline{t}, \overline{t})$. Together, these conditions imply $h(s) > 0 \; \forall \; s \in (\underline{t}, \overline{t})$, so $g > 0$ when $g' = 0$. By Lemma A3, this implies $g(s) > 0 \; \forall \; s \in (\underline{t}, \overline{t})$, so it must be the case that $R_2 > R_1$ when $\Omega = 0$. **Q.E.D.**

Proof of Proposition 4: The ranking of first-price and all-pay absolute auctions is already established in Proposition 2. Using integration by parts, the expected revenue from a second-price absolute auction is

$$R_2 = \int_{\underline{t}}^{\overline{t}} B_2(t) dF_2^n(t) = B_2(\overline{t}) - \int_{\underline{t}}^{\overline{t}} F_2^n(t) dB_2(t).$$

As $n \to \infty$, the integral on the right-hand-side converges to zero by the Lebesgue Dominated Convergence Theorem because $F_2^n(t) \to 0$ for all $t < \overline{t}$. Therefore as $n \to \infty$ $R_2 \to B_2(\overline{t}) = \frac{\overline{t}}{1-\Delta}$. Using equation (21) in the text and integration by parts, the expected revenue from an all-pay absolute auction is

$$R_A = \left(\frac{\overline{t}}{1-\theta} \right) - \left(\frac{1}{1-\theta} \right) \int_{\underline{t}}^{\overline{t}} F_1^n(t) d \left\{ t - \frac{[1 - F(t)]}{f(t)} \right\}$$

As $n \to \infty$, the integral on the right-hand side converges to zero by the Lebesgue Dominated Convergence Theorem because $F_1^n(t) \to 0$ for all $t < \overline{t}$. Hence $R_A \to \frac{\overline{t}}{1-\theta}$ as $n \to \infty$. When $\lambda > 0$, $\frac{\overline{t}}{1-\theta} > \frac{\overline{t}}{1-\Delta}$, so $R_A > R_2$ for sufficiently large n. **Q.E.D.**

B.2 The strong auctioneer case

Proof of Proposition 5: By (18) if p_g is the same in both auctions, the expected payoff functions differ (at most) by a constant. But in both auctions $U_g(\underline{t}) = 0$, because the strong auctioneer extracts all surplus from the bidder with $t = \underline{t}$. Thus the two expected payoff functions, U, are identical, *i.e.* the auctions are payoff-equivalent. It follows from equation (19) that the auctions yield the same expected revenue, *i.e.* they are also revenue-equivalent. **Q.E.D.**

Proof of Corollary 1: In both auctions the probability of type t winning is 0 for $t < \widehat{t}$ and $F^{n-1}(t)$ for $t > \widehat{t}$ and so Proposition 5 applies. **Q.E.D.**

Proof of Proposition 6: The auctioneer's revenue maximization problem is exactly the same as in the standard (non-charity) case with the sole exception that, from (19), the objective (expected revenue) is just a positive multiple of that in the standard problem since $\Omega > 0$. Thus the solution in terms of $p_g(t)$, the probability of each type t winning, is the same in both, and the requirement that $U(\underline{t}) = 0$ is implemented by the use of the surplus-extracting continuation fee. As in the non-charity case, the optimal threshold $\widehat{t}$ is the one at which the virtual valuation vanishes:

$$\widehat{t} - \frac{[1 - F(\widehat{t})]}{f(\widehat{t})} = 0.$$

Or, if the virtual valuation is positive for all types, $\widehat{t} = \underline{t}$. **Q.E.D.**

B.3 The weak auctioneer case

Proof of Proposition 7: In a first-price auction, equation (6) characterizes a binding $\widehat{t}$. This provides us with the following relationship between r and φ, holding $\widehat{t}$ fixed:

$$\frac{dr}{d\varphi} = \frac{-1}{F(\widehat{t})^{n-1}}.$$

In a first-price auction, $R_1 = nE_t[x_1(t|r,\varphi)]$ with $x_1(t|r,\varphi) = \varphi + B_1(t|r,\varphi)F(t)^{n-1}$ for bidders with $t \geq \widehat{t}$ and $x_1(t|r,\varphi) = 0$ otherwise. For the threshold bidder, $\frac{dx_1(\widehat{t}|r,\varphi)}{d\varphi} = 0$ since $B_1(\widehat{t}|r,\varphi) = r$ and $\frac{dr}{d\varphi} = -F(\widehat{t})^{1-n}$. However, for bidders with $t > \widehat{t}$

$$\frac{dx_1(t|r,\varphi)}{d\varphi} = 1 - \left(\frac{F(\widehat{t})}{F(t)}\right)^{\alpha-n+1} > 0,$$

so $\frac{dR_1}{d\varphi} > 0$ for any $\widehat{t} \in (\underline{t}, \overline{t})$. Thus, for a given $\widehat{t}$ in a first-price auction, the weak auctioneer's revenue is maximized when he sets $r = 0$ and $\varphi > 0$.

In a second-price auction, the binding $\widehat{t}$ is characterized by equation (11). The following expression describes how r changes with φ to keep $\widehat{t}$ constant:

$$\frac{dr}{d\varphi} = \frac{-1}{F(\widehat{t})^{n-1} + \frac{\lambda(n-1)}{(1-\theta)}F(\widehat{t})^{n-2}[1 - F(\widehat{t})]}.$$

Expected revenue in a second-price auction is $R_2 = nE_t[x_2(t|r,\varphi)]$, with

$$x_2(t|r,\varphi) = \varphi + F(\widehat{t})^{n-1}r + \int_{\widehat{t}}^{t} B_2(t)dF_1^{n-1}(t)$$

for bidders with $t \geq \widehat{t}$, and $x_2(t|r,\varphi) = 0$ for bidders with $t < \widehat{t}$. For active bidders, the effect of a change in φ on x_2 is

$$\frac{dx_2(t|r,\varphi)}{d\varphi} = 1 - \left\{\frac{F(\widehat{t})^{n-1}}{F(\widehat{t})^{n-1} + \frac{\lambda(n-1)}{(1-\theta)}F(\widehat{t})^{n-2}[1 - F(\widehat{t})]}\right\}.$$

The fraction in braces is always in $(0,1)$ for $\widehat{t} \in (\underline{t}, \overline{t})$, so $\frac{dx_2(t)}{d\varphi} > 0$ and $\frac{dR_2}{d\varphi} > 0$ for a fixed $\widehat{t}$. Thus, in the second-price auction it is best for the revenue-maximizing weak auctioneer to set $r = 0$ and $\varphi > 0$ for a given $\widehat{t}$.

Finally, in an all-pay auction the indifference condition that determines $\widehat{t}$ is (15). Therefore, $\frac{dr}{d\varphi} = -1$. The expected payment of each active bidder is $x_A(t) = \varphi + B_A(t|r,\varphi)$, and $R_A = nE_t[x_A(t|r,\varphi)]$. The bid function B_A has $\frac{\partial B_A}{\partial r} = 1$, and this paired with $\frac{dr}{d\varphi} = -1$ to yield $\frac{dx_A}{d\varphi} = 0$. Thus, the weak auctioneer is indifferent between using r and φ to induce a threshold bidder type of $\widehat{t}$. **Q.E.D.**

Proof of Proposition 8: When an auction of format g induces a threshold type of $\widehat{t}$, equation (20) provides the relationship between auction revenue and bidder utility. For each auction format, we describe a relationship between $U_g(\widehat{t})$ and X_g. Fix a value of $\widehat{t}$ and let φ_g be the value of the bidding fee that yields this $\widehat{t}$. (Recall $r = 0$ following Proposition 7.) Across auction formats, the *ex ante* expected payments of bidders are

$$\begin{aligned}
X_1 &= [1 - F(\widehat{t})]\varphi_1 + \int_{\widehat{t}}^{\overline{t}} B_1(t|r,\varphi) F^{n-1}(t) f(t) dt \\
X_2 &= [1 - F(\widehat{t})]\varphi_2 + F(\widehat{t})^{n-1}[1 - F(\widehat{t})]r + \int_{\widehat{t}}^{\overline{t}} B_2(t)(n-1)F(t)^{n-2}[1 - F(t)] f(t) dt \\
X_A &= [1 - F(\widehat{t})]\varphi_A + \int_{\widehat{t}}^{\overline{t}} B_A(t|r,\varphi) f(t) dt.
\end{aligned}$$

We use the form of X_g in each auction to construct an expression $nU_g(\widehat{t}) = n\lambda(n-1)X_g + \lambda\delta_g$ which allows us to simplify equation (20). The auctions' values of $U_g(\widehat{t})$ are given in Section 3. Solving $\lambda\delta_g = nU_g(\widehat{t}) - n\lambda(n-1)X_g$ yields $\delta_A = 0$ and

$$\begin{aligned}
\delta_1 &= \int_{\widehat{t}}^{\overline{t}} B_1(t|r,\varphi)[ndF_1^{n-1}(t) - (n-1)dF_1^n(t)] = \int_{\widehat{t}}^{\overline{t}} B_1(t|r,\varphi) dF_2^n(t) \\
\delta_2 &= \int_{\widehat{t}}^{\overline{t}} B_2(t)[ndF_2^{n-1}(t) - (n-1)dF_2^n(t)].
\end{aligned}$$

Next, we cancel the terms $n\lambda(n-1)X_g$ from each side of equation (20), which allows us to write the expected revenue of each auction as

$$R_g = \left(\frac{1}{1-\theta}\right) \int_{\widehat{t}}^{\overline{t}} \left\{ t - \frac{[1 - F(t)]}{f(t)} \right\} dF_1^n(t) - \frac{\lambda\delta_g}{1-\theta}.$$

Since, $\delta_A = 0$, it is immediately apparent that R_A is bounded. δ_1 is constructed from the bounded functions $B_1(t|r,\varphi)$ and $\frac{dF_2^n(t)}{dt}$, and δ_2 is constructed from the bounded bidding function B_2 and a weighted combination of two bounded density functions. Thus, δ_1 and δ_2 are themselves bounded, which implies that R_1 and R_2 are bounded. **Q.E.D.**

Proof of Corollary 2: The revenue ranking $R_A > R_1$ follows immediately from the sign of δ_1,

$$\delta_1 = \int_{\widehat{t}}^{\overline{t}} B_1(t|r,\varphi)dF_2^n(t) > 0.$$

When $n = 2$ we have

$$\delta_2 = -\int_{\widehat{t}}^{\overline{t}} B_2(t)dF_2^2(t) < 0,$$

so $R_2 > R_A$.

To evaluate δ_1 and δ_2 when $n \to \infty$, let $G_1(t) = F_2^n(t)$ and $G_2(t) = \{nF_2^{n-1}(t) - (n-1)F_2^n(t)\}$, and write

$$\delta_g = \int_{\widehat{t}}^{\overline{t}} B_g(t|r,\varphi)dG_g(t)$$

for $g \in \{1, 2\}$. Using integration by parts, we rewrite δ_g as

$$\delta_g = B_g(\overline{t}|r,\varphi) - B_g(\widehat{t}|r,\varphi)G_g(\widehat{t}) - \int_{\widehat{t}}^{\overline{t}} G_g(t)dB_g(t|r,\varphi).$$

$G_g(t) \to 0$ as $n \to \infty$ for any $t < \overline{t}$. B_2 is unaffected by n, while B_1 approaches $\frac{t}{1-\Delta}$ as $n \to \infty$ and $\frac{dB_1(t|r,\varphi)}{dt}$ is bounded in the limit. These characteristics of G_g and B_g imply that the integral in the expression above vanishes as $n \to \infty$ by the Lebesgue Dominated Convergence Theorem, and $B_g(\widehat{t}|r,\varphi)G_g(\widehat{t})$ converges to zero because $\widehat{t} < \overline{t}$. Thus, $\delta_1 \to B_1(\overline{t}|r,\varphi)$ and $\delta_2 \to B_2(\overline{t})$ as $n \to \infty$. The sign of δ_2 in the limit implies $R_A > R_2$ for a sufficiently large n, and $B_1(\overline{t}|r,\varphi) = B_2(\overline{t})$ as $n \to \infty$ so R_2 and R_1 converge. **Q.E.D.**

Proof of Proposition 9: We begin with the relatively simple case of revenue maximization in an all-pay auction. The weak auctioneer will set φ_A (or r_A) to implicitly select a revenue-maximizing value of $\widehat{t}$. If $\underline{t} < f(\underline{t})^{-1}$, then revenue is maximized with the familiar condition

$$\widehat{t} - \frac{[1 - F(\widehat{t})]}{f(\widehat{t})} = 0.$$

Otherwise $\widehat{t} = \underline{t}$.

In a first-price auction, the effect of a change in $\widehat{t}$ on revenue is

$$\frac{dR_1}{d\widehat{t}} = \left(\frac{-1}{1-\theta}\right)\left\{\widehat{t} - \frac{[1 - F(\widehat{t})]}{f(\widehat{t})}\right\}\frac{dF_1^n(\widehat{t})}{d\widehat{t}} - \frac{\lambda}{1-\theta}\left\{-B_1(\widehat{t}|0,\varphi) + \int_{\widehat{t}}^{\overline{t}} \frac{\partial B_1(t|0,\varphi)}{\partial \widehat{t}}dF_2^n(t)\right\}.$$

$B_1(\widehat{t}|0,\varphi) = 0$ when $r = 0$, and

$$\frac{\partial B_1(t|0,\varphi)}{\partial \widehat{t}} = \left(\frac{-1}{1-\Delta}\right)\left(\frac{\widehat{t}}{F(t)^\alpha}\right)\frac{dF(\widehat{t})^\alpha}{d\widehat{t}} < 0.$$

Since $\int_{\widehat{t}}^{\bar{t}} \frac{\partial B_1(t|0,\varphi)}{\partial \widehat{t}} dF_2^n(t) < 0$, $\frac{d\delta_1}{d\widehat{t}} < 0$. Therefore, $\frac{dR_1}{d\widehat{t}} > \frac{dR_A}{d\widehat{t}}$, and when there exists a value of $\widehat{t}_1$ such that $\frac{dR_1}{d\widehat{t}} = 0$ it will be greater than the the optimal threshold for an all-pay auction, $\widehat{t}_A$, at which $\frac{dR_A}{d\widehat{t}} = 0$ (or $\widehat{t}_A = \underline{t}$). Otherwise $\widehat{t}_1 = \underline{t}$.

In a second-price auction with $n = 2$, $\delta_2 = -\int_{\widehat{t}}^{\bar{t}} B_2(t)dF_2^2(t)$, so

$$R_2(\widehat{t}) = R_A(\widehat{t}) + \frac{\lambda}{1-\theta}\int_{\widehat{t}}^{\bar{t}} B_2(t)dF_2^2(t).$$

$R_2'(\widehat{t}) = R_A'(\widehat{t}) - \frac{\lambda}{1-\theta}B_2(\widehat{t})dF_2^2(\widehat{t})$, so $R_2'(\widehat{t}) < R_A'(\widehat{t})$. This leads the weak auctioneer to select a lower value of $\widehat{t}$ for a second-price auction than in an all-pay auction, provided that the all-pay solution is greater than $\underline{t}$. When $n > 2$ and

$$R_2(\widehat{t}) = R_A(\widehat{t}) - \frac{\lambda\delta_2}{1-\theta},$$

we have

$$R_2'(\widehat{t}) = R_A'(\widehat{t}) + \frac{\lambda}{1-\theta}B_2(\widehat{t})\left[1 - \left(\frac{n-1}{n-2}\right)F(\widehat{t})\right] n\frac{dF_2^{n-1}(\widehat{t})}{dt}.$$

Fix $\widehat{t}$ at the revenue-maximizing value for an all-pay auction, and define $y = F^{-1}(\frac{n-2}{n-1})$. Since y tends to $\bar{t}$ as $n \to \infty$, there must exist a finite $\widetilde{n}$ such that $y > \widehat{t}$ for all $n > \widetilde{n}$. Whenever $n > \widetilde{n}$, $R_2'(\widehat{t}) > R_A'(\widehat{t})$ and the auctioneer chooses a greater threshold in the second-price auction than in the all-pay auction. **Q.E.D.**

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